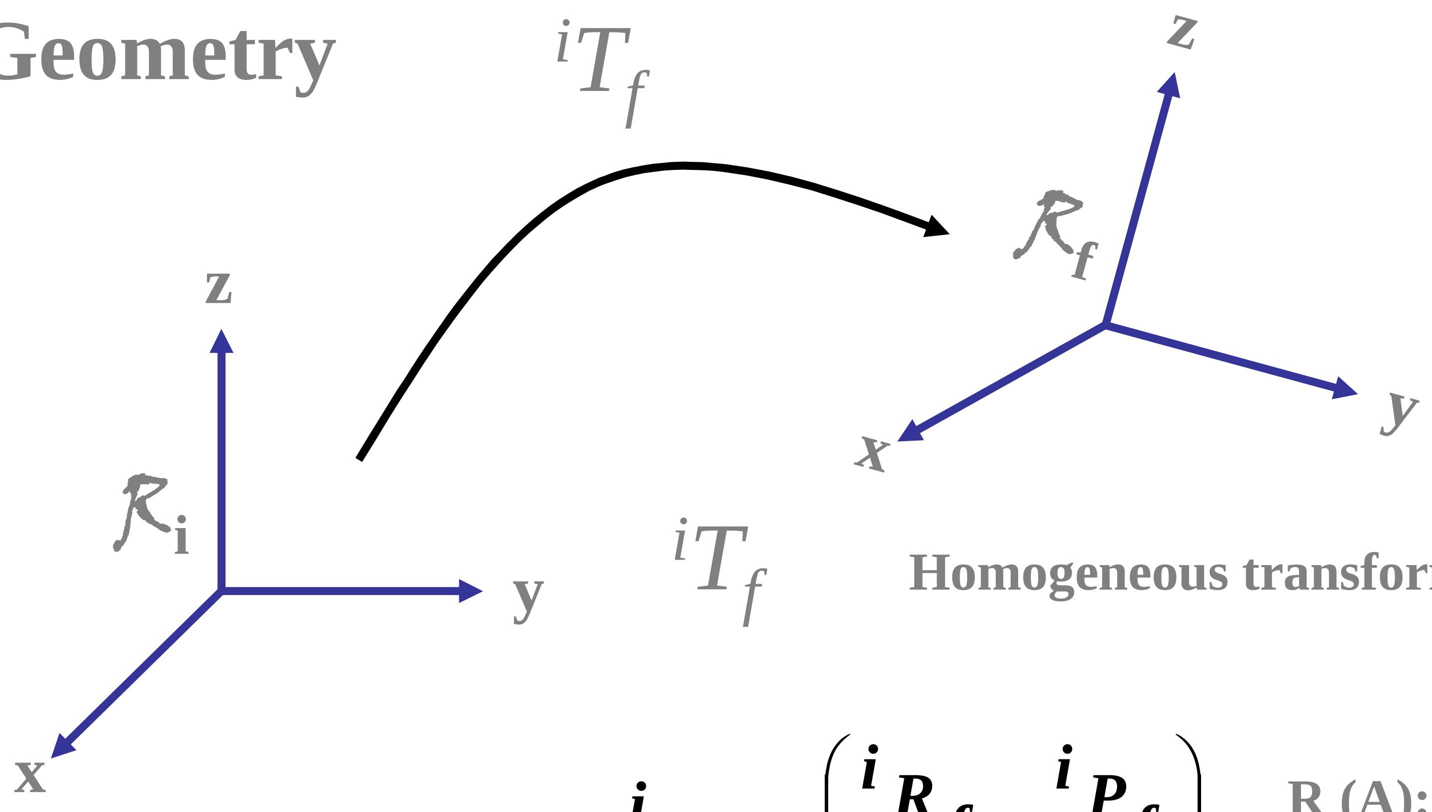
Basic Knowledge *Homogeneous transformation*

Geometry



$^{i}T_{f}$ **Homogeneous transformation matrix**

$$^{i}\boldsymbol{T}_{\boldsymbol{f}} = \begin{pmatrix} ^{i}\boldsymbol{R}_{\boldsymbol{f}} & ^{i}\boldsymbol{P}_{\boldsymbol{f}} \\ \boldsymbol{000} & \boldsymbol{1} \end{pmatrix}$$

R (A): Orientation
P : Position

Basic Knowledge *Homogeneous transformation*

Geometry

Consider a 3D point in space $\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{R_f}$

Then

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{R_i} = {}^{i}T_f \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{R_f} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{R_i} = {}^{i}R_f \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{R_f} + {}^{i}P_f$$

Basic Knowledge *Homogeneous transformation*

Geometry $^{i}T_{f}$ **Homogeneous transformation matrix**

Different representations

R : Orientation

s, n, a Cosinus directors
RPY angles (Roll (z), Pitch(y), Yaw(x))
Briant angles (x,y,z)
Euler angles (z,x,z)
$\underline{u}.\theta$, $\underline{u}.\sin(\theta)$, $\underline{u}.\sin(\theta/2)$,
Quaternion $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

P : Position

Cartesian coordinates
Cylindrical coordinates
Spherical coordinates

Basic Knowledge *Homogeneous transformation*

Geometry $^{i}T_{f}$ Homogeneous transformation matrix

Different representations (i.e)

R : Orientation

s, n, a Director Cosinus

$$R = \begin{pmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{pmatrix}$$

P : Position

Cartesian coordinates

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P_x} \\ \boldsymbol{P_y} \\ \boldsymbol{P_z} \end{pmatrix}$$

No rotation
$R = I_3$

No translation
P=(0,0,0)^T

Basic Knowledge *Homogeneous transformation*

Geometry $\;{}^{i}T_{f}\;$ Homogeneous transformation matrix

Main properties of the rotation matrix

R : Orientation

s, n, a Director Cosinus

$$R=\begin{pmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{pmatrix}$$

$$\|\underline{s}\| = \|\underline{n}\| = \|\underline{a}\| = 1$$

$$\underline{s}\cdot\underline{n} = 0 \qquad \underline{s}\times\underline{n} = \underline{a}$$

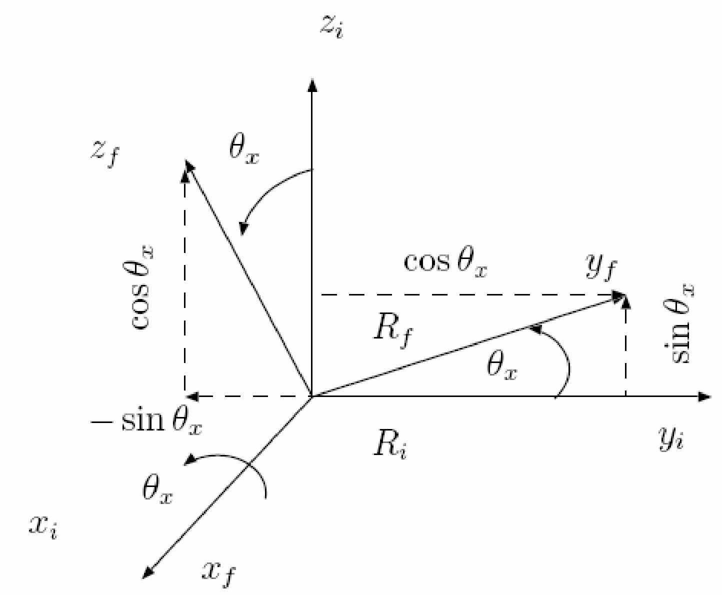
$$\underline{s}\cdot\underline{a} = 0 \qquad \underline{n}\times\underline{a} = \underline{s}$$

$$\underline{a}\cdot\underline{n} = 0 \qquad \underline{a}\times\underline{s} = \underline{n}$$

$$R^{-1} = R^{T}$$

Basic Knowledge *Homogeneous transformation: Rotation matrix*

Geometry $\quad {}^{i}R_{f}$



Rotation matrix Rot(x,θ_x)

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\theta_x & -\mathrm{s}\theta_x \\ 0 & \mathrm{s}\theta_x & \mathrm{c}\theta_x \end{pmatrix}$$

$$\begin{array}{lllllll} \underline{i_{R_f}} & = 1.\underline{i_{R_i}} & + & 0.\underline{j_{R_i}} & + & 0.\underline{k_{R_i}} & = {}^{i}\underline{s}_f \\ \underline{j_{R_f}} & = 0.\underline{i_{R_i}} & + & \cos\theta_x.\underline{j_{R_i}} & + & \sin\theta_x.\underline{k_{R_i}} & = {}^{i}\underline{n}_f \\ \underline{k_{R_f}} & = 0.\underline{i_{R_i}} & - & \sin\theta_x.\underline{j_{R_i}} & + & \cos\theta_x.\underline{k_{R_i}} & = {}^{i}\underline{a}_f \end{array}$$

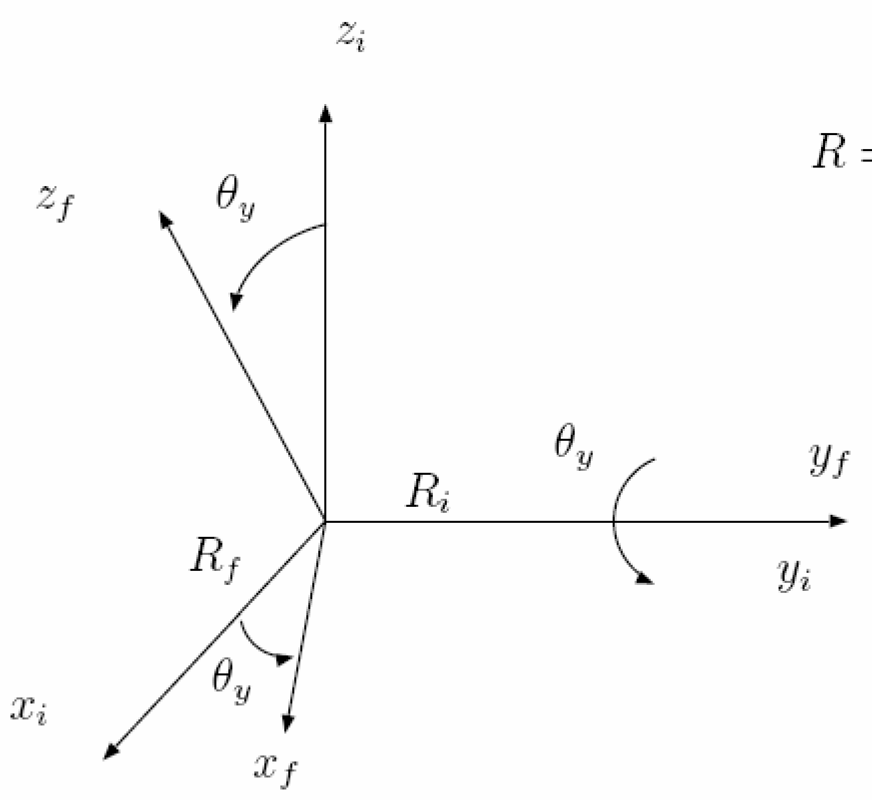
$$\Rightarrow \quad \begin{array}{ll} \underline{i_{R_i}} & = {}^{f}\underline{s}_i \\ \underline{j_{R_i}} & = {}^{f}\underline{n}_i \\ \underline{k_{R_i}} & = {}^{f}\underline{a}_i \end{array}$$

Matrix to change the frame for one vector

Basic Knowledge *Homogeneous transformation: Rotation matrix*

Geometry $^{i}R_{f}$

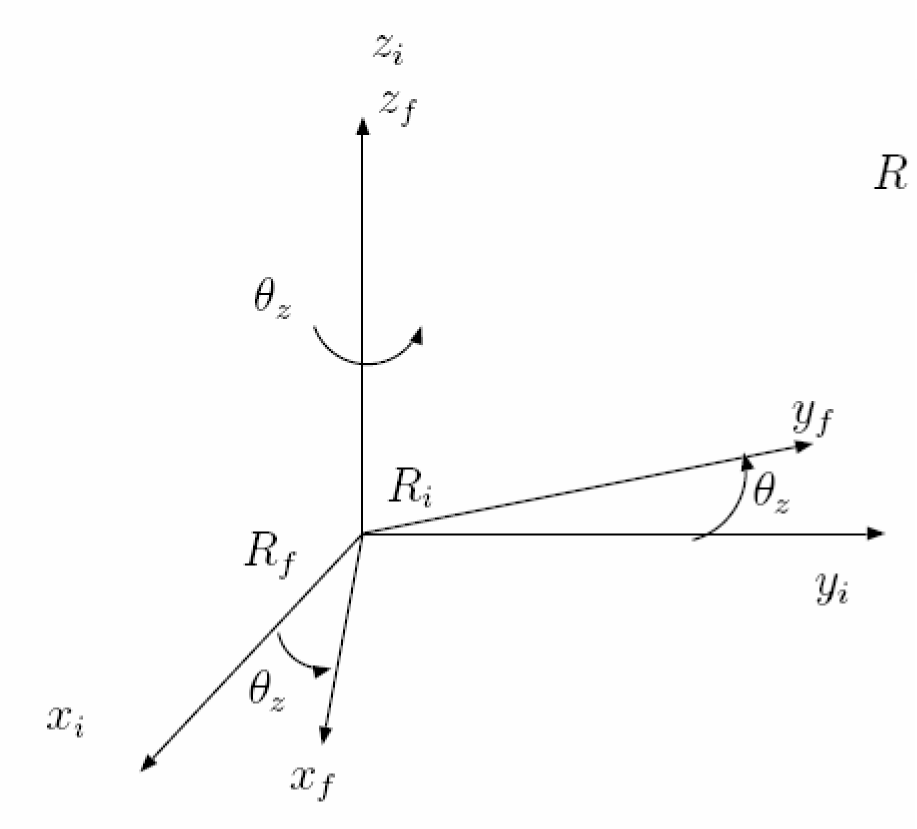
Rotation matrix Rot(y,θ_y)



$$R = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} = \begin{pmatrix} c\theta_y & 0 & s\theta_y \\ 0 & 1 & 0 \\ -s\theta_y & 0 & c\theta_y \end{pmatrix}$$

Basic Knowledge *Homogeneous transformation: Rotation matrix*

Geometry $^{i}R_{f}$ Rotation matrix Rot(z,θ_z)

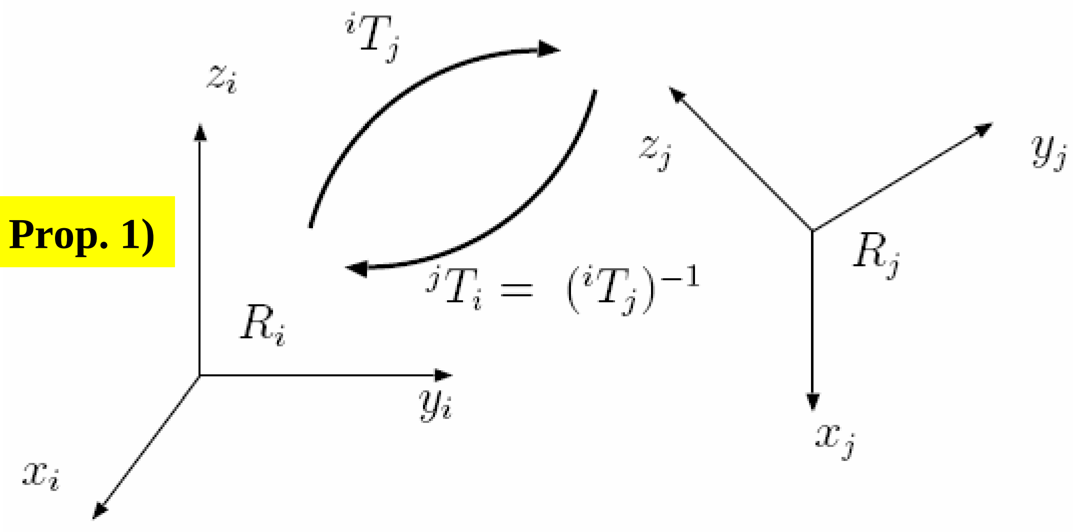


$$R = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\theta_z & -s\theta_z & 0 \\ s\theta_z & c\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Basic Knowledge *Homogeneous transformation properties*

Geometry $^{i}T_{f}$ **Homogeneous transformation matrix**

Prop. 1)



$$\boxed{({}^{i}T_j)^{-1} = {}^{j}T_i}$$

Prop. 2)

$$\boxed{(T)^{-1} = (\mathrm{Rot}(\underline{u}, \theta_u))^{-1} = \mathrm{Rot}(\underline{u}, -\theta_u) = \mathrm{Rot}(-\underline{u}, \theta_u)}$$

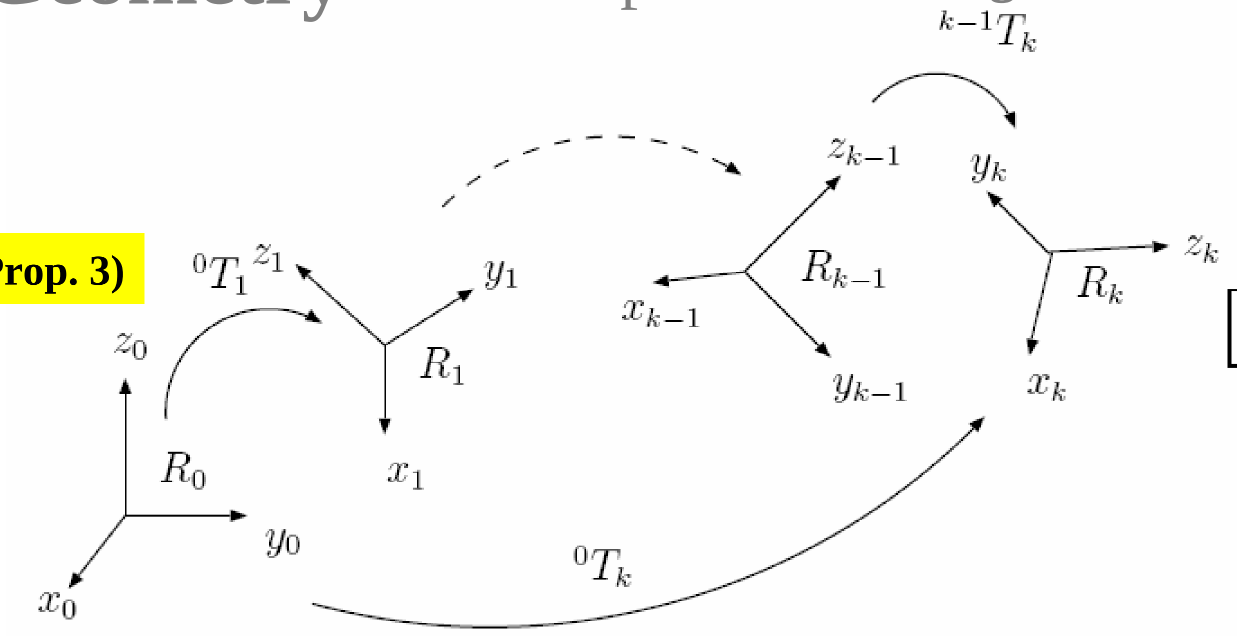
$$\mathrm{Rot}(x, -\theta_x) = (\mathrm{Rot}(x, \theta_x))^T = (\mathrm{Rot}(x, \theta_x))^{-1}$$

$$(\mathrm{Trans}(u, d))^{-1} = \mathrm{Trans}(-u, d) = \mathrm{Trans}(u, -d)$$

Basic Knowledge *Homogeneous transformation properties*

Geometry $^{i}T_{f}$ Homogeneous transformation matrix

Prop. 3)



$$\boxed{^{0}T_k = {}^{0}T_1 . {}^{1}T_2 . {}^{2}T_3 . {}^{3}T_4 \ldots {}^{k-1}T_k}$$

Prop. 4)

$$\boxed{T^{-1} = \begin{pmatrix} & -\underline{s}^T.P \\ A^T & -\underline{n}^T.P \\ & -\underline{a}^T.P \\ 0\ 0\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^T & -A^T.P \\ 0\ 0\ 0 & 1 \end{pmatrix}}$$

Basic Knowledge *Homogeneous transformation properties*

Geometry $\quad {}^{i}T_{f} \quad$ **Homogeneous transformation matrix**

Prop. 5)

$$T_1.T_2 = \begin{pmatrix} A_1 & P_1 \\ 0\,0\,0 & 1 \end{pmatrix} . \begin{pmatrix} A_2 & P_2 \\ 0\,0\,0 & 1 \end{pmatrix} = \begin{pmatrix} A_1.A_2 & A_1.P_2 + P_1 \\ 0\,0\,0 & 1 \end{pmatrix}$$

$$T_2.T_1 = \begin{pmatrix} A_2 & P_2 \\ 0\,0\,0 & 1 \end{pmatrix} . \begin{pmatrix} A_1 & P_1 \\ 0\,0\,0 & 1 \end{pmatrix} = \begin{pmatrix} A_2.A_1 & A_2.P_1 + P_2 \\ 0\,0\,0 & 1 \end{pmatrix}$$

$$\boxed{T_1.T_2 \neq T_2.T_1}$$

Prop. 6)

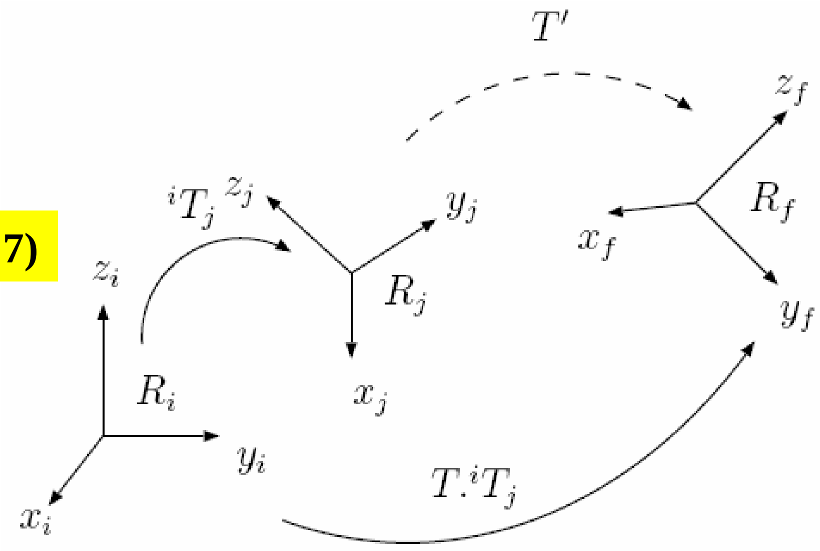
$$\boxed{\mathrm{Rot}(\underline{u}, \theta_1).\mathrm{Rot}(\underline{u}, \theta_2) = \mathrm{Rot}(\underline{u}, \theta_1 + \theta_2)}$$

$$\boxed{\mathrm{Trans}(\underline{u}, d).\mathrm{Rot}(\underline{u}, \theta_1) = \mathrm{Rot}(\underline{u}, \theta_1).\mathrm{Trans}(\underline{u}, d)}$$

Basic Knowledge *Homogeneous transformation properties*

Geometry $^{i}T_{f}$ Homogeneous transformation matrix

Prop. 7)



$$\boxed{{}^{i}T_f = T.{}^{i}T_j}$$

T is defined in R_i

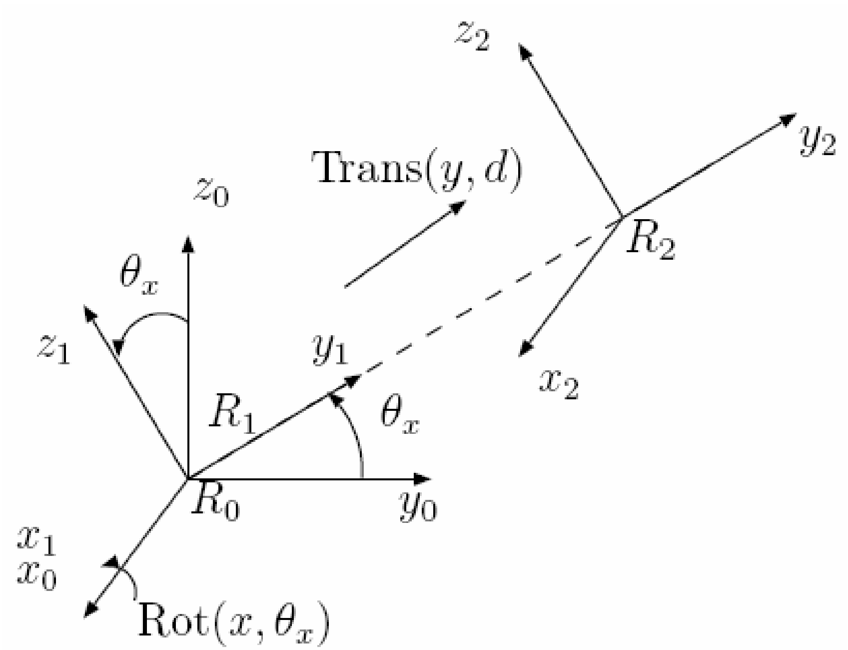
$$\boxed{{}^{i}T_f = {}^{i}T_j.T'}$$

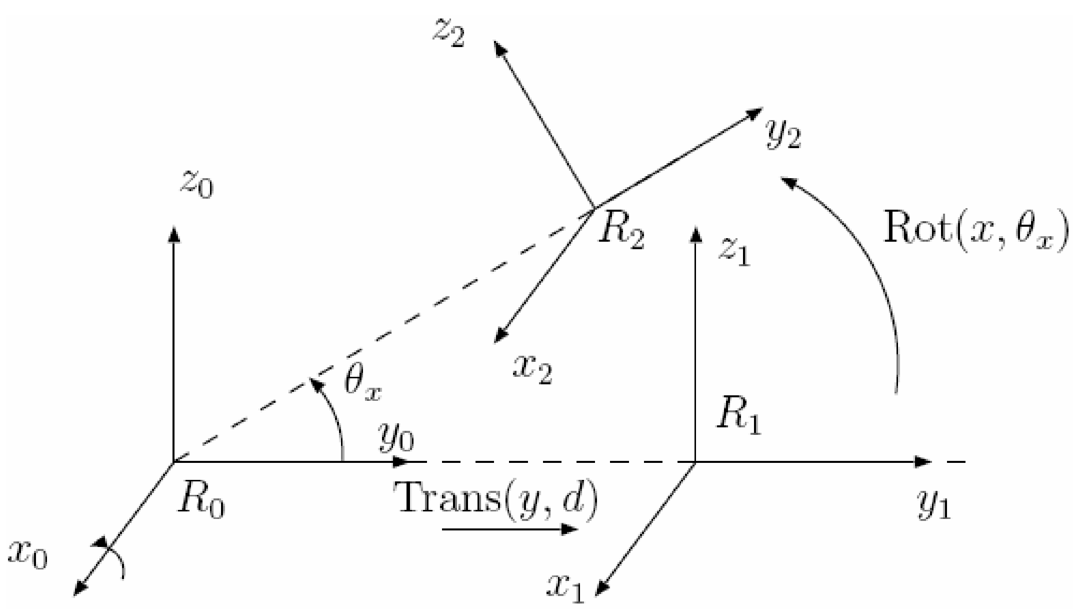
T' is defined in R_j

$$\boxed{T.{}^{i}T_j = {}^{i}T_j.T'}$$

Basic Knowledge *Homogeneous transformation properties*

Geometry $^{i}T_{f}$ **Homogeneous transformation matrix**





Basic Knowledge *Homogeneous transformation properties*

Geometry $\;{}^{i}T_{f}\;$ Homogeneous transformation matrix

Prop. 8)

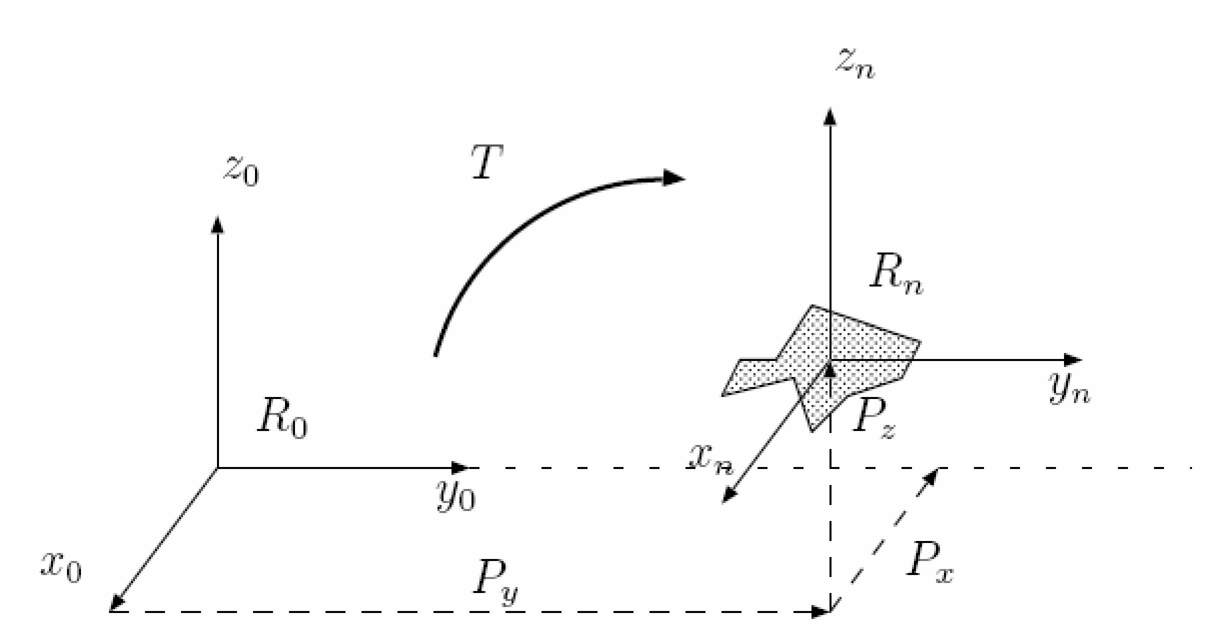
$$\hat{u} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\underline{u} \wedge \underline{V} = \hat{u}.\underline{V}$$

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \wedge \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} u_y.V_z - u_z.V_y \\ u_z.V_x - u_x.V_z \\ u_x.V_y - u_y.V_x \end{pmatrix} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} . \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Basic Knowledge *Rigid body pose parameterization: position*

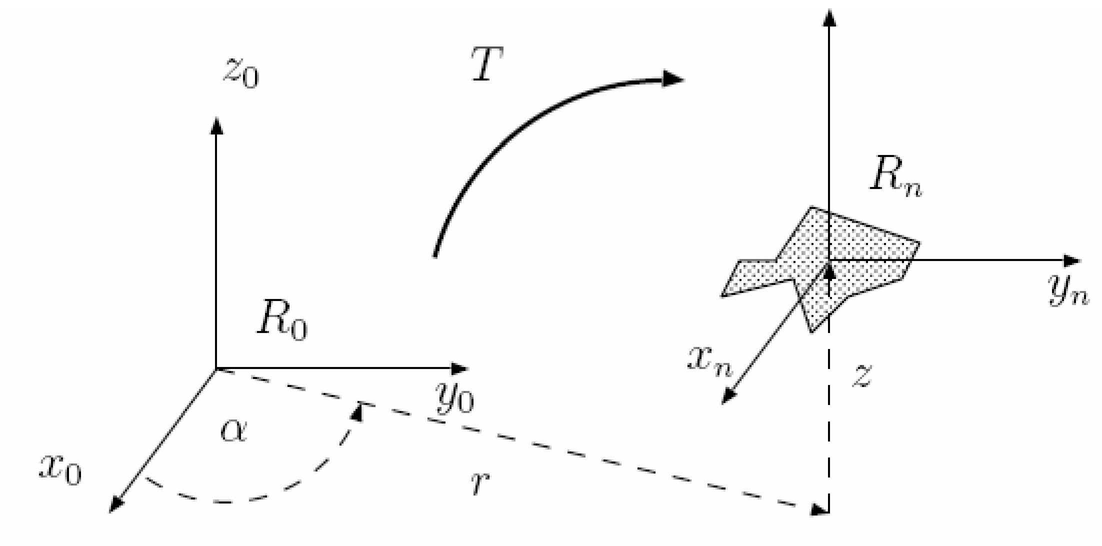
Cartesian coordinates



$$P_{\text{car}} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

Basic Knowledge *Rigid body pose parameterization: position*

Cylindrical coordinates



$$P_{\text{cyl}} = \begin{bmatrix} r.c\alpha \\ rs\alpha \\ z \end{bmatrix}$$

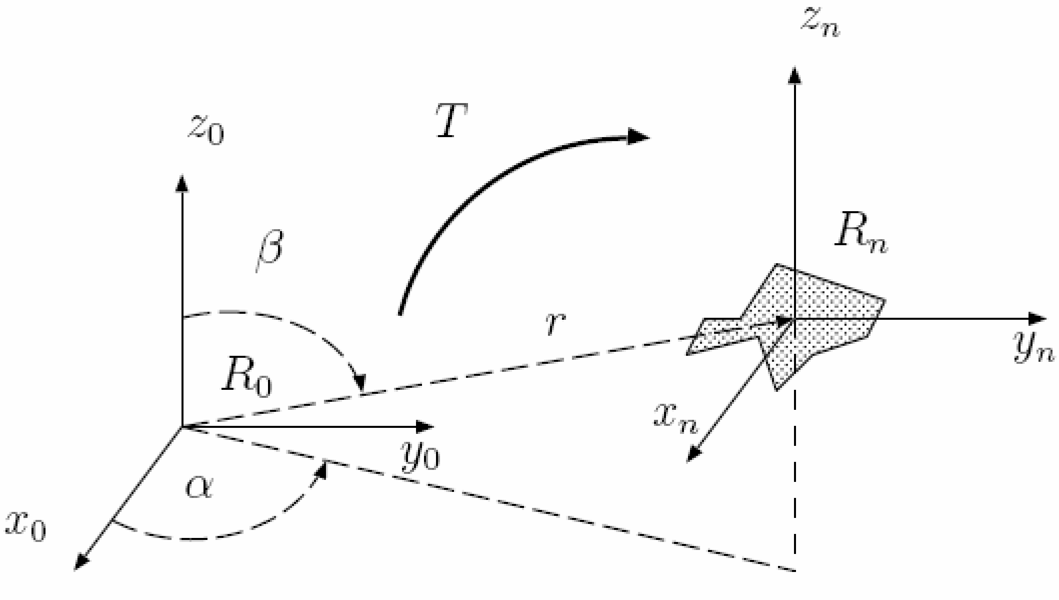
$$r = \sqrt{P_x^2 + P_y^2}$$
$$\alpha = \text{atan2}(P_y, P_x)$$
$$z = P_z$$

$$\alpha \in [-180; 180]$$

Singularity : Px=Py=0

Basic Knowledge *Rigid body pose parameterization: position*

Spherical coordinates



$$P_{\text{sph}} = \begin{bmatrix} r.c\alpha.s\beta \\ r.s\alpha.s\beta \\ r.c\beta \end{bmatrix}$$

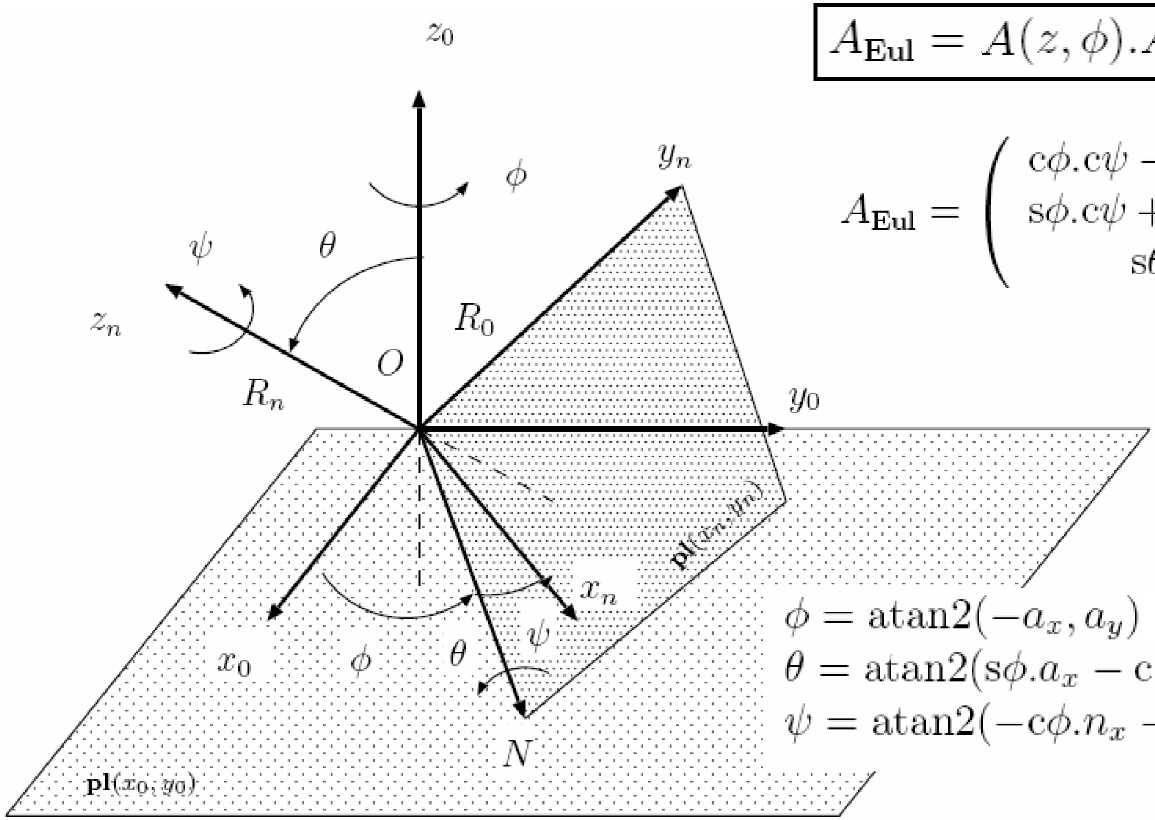
$$r = \sqrt{P_x^2 + P_y^2 + P_z^2}$$

$$\alpha = \text{atan2}(P_y, P_x) \text{ if } \beta \neq 0 \text{ or } \alpha = 0 \text{ if } \beta = 0$$

$$\beta = \text{atan2}\left(\frac{P_y}{s\alpha}, P_z\right) \text{ if } \alpha \neq 0 \text{ or } \beta = \text{atan2}(P_x, P_z) \text{ if } \alpha = 0$$

Basic Knowledge *Rigid body pose parameterization: orientation*

Euler angles (z, x, z)



$$\boxed{A_{\text{Eul}} = A(z,\phi).A(x,\theta).A(z,\psi)}$$

$$A_{\text{Eul}} = \begin{pmatrix} c\phi.c\psi - s\phi.c\theta.s\psi & -c\phi.s\psi - s\phi.c\theta.c\psi & s\phi.s\theta \\ s\phi.c\psi + c\phi.c\theta.s\psi & -s\phi.s\psi + c\phi.c\theta.c\psi & -c\phi.s\theta \\ s\theta.s\psi & s\theta.c\psi & c\theta \end{pmatrix}$$

$\phi = \text{atan2}(-a_x, a_y)$ or $(\phi = \text{atan2}(a_x, -a_y) + 180^0)$

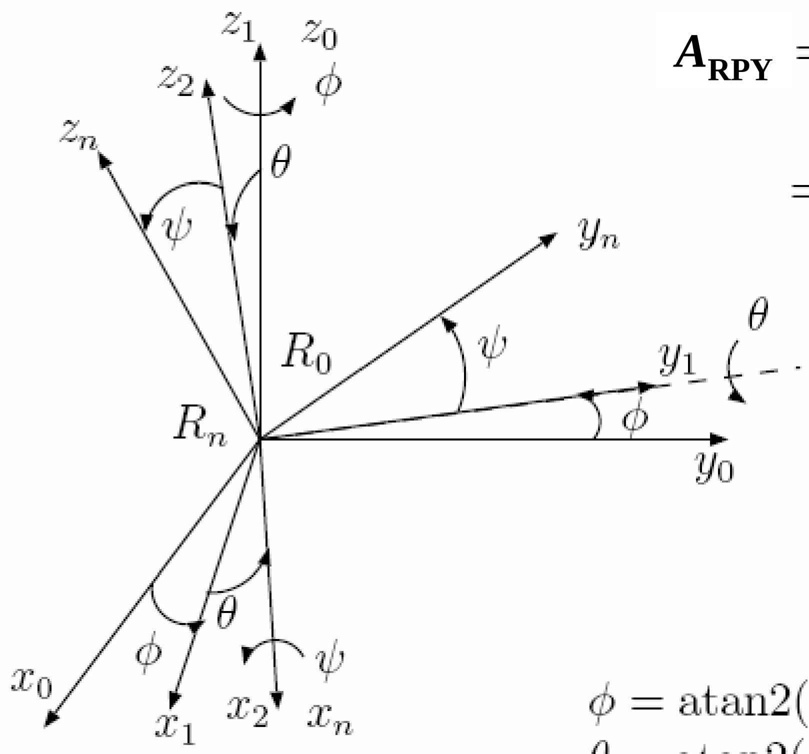
$\theta = \text{atan2}(s\phi.a_x - c\phi.a_y, a_z)$

$\psi = \text{atan2}(-c\phi.n_x - s\phi.n_y, c\phi.s_x + s\phi.s_y)$

Remarks: Euler angles can be defined also by (z,y,z) ➔ $A_{\text{Eul}} = A(z,\phi).A(y,\theta).A(z,\psi)$

Basic Knowledge *Rigid body pose parameterization: orientation*

RPY angles (z,y,x)



$$\boldsymbol{A}_{\mathbf{RPY}} = A(z,\phi).A(y,\theta).A(x,\psi)$$

$$= \begin{pmatrix} \mathrm{c}\phi.\mathrm{c}\theta & \mathrm{c}\phi.\mathrm{s}\theta.\mathrm{s}\psi - \mathrm{s}\phi.\mathrm{c}\psi & \mathrm{c}\phi.\mathrm{s}\theta.\mathrm{c}\psi + \mathrm{s}\phi.\mathrm{s}\psi \\ \mathrm{s}\phi.\mathrm{c}\theta & \mathrm{s}\phi.\mathrm{s}\theta.\mathrm{s}\psi + \mathrm{c}\phi.\mathrm{c}\psi & \mathrm{s}\phi.\mathrm{s}\theta.\mathrm{c}\psi - \mathrm{c}\phi.\mathrm{s}\psi \\ -\mathrm{s}\theta & \mathrm{c}\theta.\mathrm{s}\psi & \mathrm{c}\theta.\mathrm{c}\psi \end{pmatrix}$$

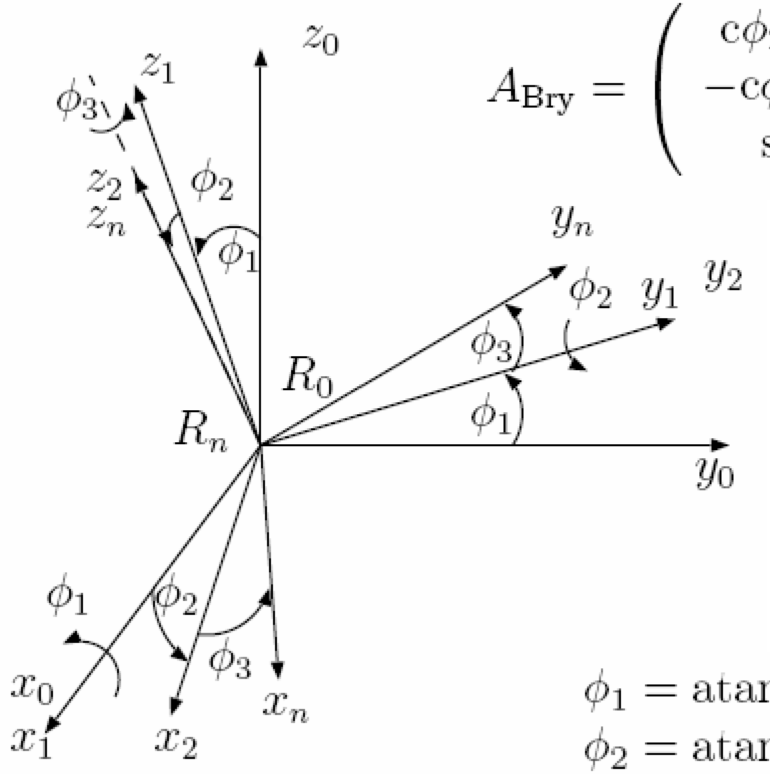
$\phi = \mathrm{atan2}(s_y, s_x)$ or $(\phi = \mathrm{atan2}(-s_y, -s_x) + 180^0)$

$\theta = \mathrm{atan2}(-s_z, \mathrm{c}\phi.s_x + \mathrm{s}\phi.s_y)$

$\psi = \mathrm{atan2}(\mathrm{s}\phi.a_x - \mathrm{c}\phi.a_y, -\mathrm{s}\phi.n_x + \mathrm{c}\phi.n_y)$

Basic Knowledge *Rigid body pose parameterization: orientation*

Bryant angles (x,y,z) $\boxed{A_{\text{Bry}} = A(x,\phi_1).A(y,\phi_2).A(z,\phi_3)}$



$$A_{\text{Bry}} = \begin{pmatrix} c\phi_2.c\phi_3 & c\phi_1.s\phi_3 + s\phi_1.s\phi_2.c\phi_3 & s\phi_1.s\phi_3 - c\phi_1.s\phi_2.c\phi_3 \\ -c\phi_2.s\phi_3 & c\phi_1.c\phi_3 - s\phi_1.s\phi_2.s\phi_3 & s\phi_1.c\phi_3 + c\phi_1.s\phi_2.s\phi_3 \\ s\phi_2 & -s\phi_1.c\phi_2 & c\phi_1.c\phi_2 \end{pmatrix}$$

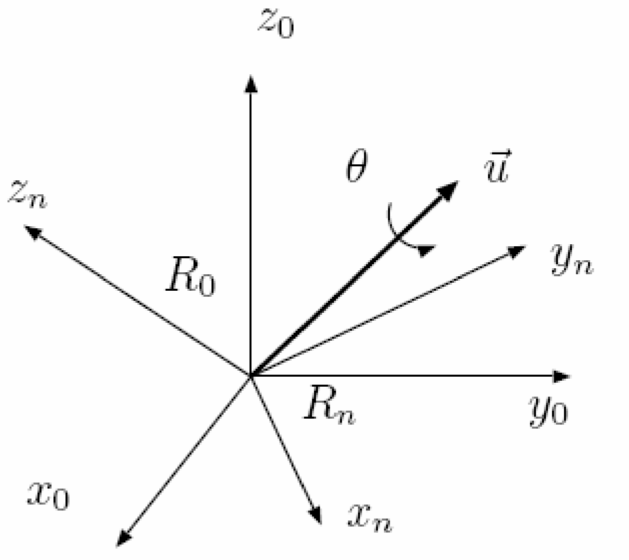
$\phi_1 = \text{atan2}(-n_z, a_z)$ or $(\phi_1 = \text{atan2}(n_z, -a_z) + 180^0)$

$\phi_2 = \text{atan2}(s_z, c\phi_1.a_z - s\phi_1.n_z)$

$\phi_3 = \text{atan2}(c\phi_1.n_x + s\phi_1.a_x, c\phi_1.n_y + s\phi_1.a_y)$

Basic Knowledge *Rigid body pose parameterization: orientation*

Orientation ($\underline{u}$,θ)



$$A(\underline{u},\theta) = \begin{pmatrix} u_x^2.(1-c\theta)+c\theta & u_xu_y.(1-c\theta)-u_z.s\theta & u_xu_z.(1-c\theta)+u_y.s\theta \\ u_xu_y.(1-c\theta)+u_z.s\theta & u_y^2.(1-c\theta)+c\theta & u_yu_z.(1-c\theta)-u_x.s\theta \\ u_xu_z.(1-c\theta)-u_y.s\theta & u_yu_z.(1-c\theta)+u_x.s\theta & u_z^2.(1-c\theta)+c\theta \end{pmatrix}$$

$$A(\underline{u},\theta) = \underline{u}.\underline{u}^T.(1-c\theta)+I_3.c\theta+\hat{u}.s\theta$$ ***Rodrigues formula***

$$A(\underline{u},\theta) = I_3+\hat{u}.s\theta+\hat{u}^2.(1-c\theta)$$

Basic Knowledge *Rigid body pose parameterization: orientation*

Orientation ($\underline{u}$.θ)

$$A(\underline{u},\theta) = \underline{u}.\underline{u}^T.(1-c\theta) + I_3.c\theta + \hat{u}.s\theta$$

Rodrigues formula

$$A(\underline{u},\theta) = I_3 + \hat{u}.s\theta + \hat{u}^2.(1-c\theta)$$

$$\hat{u} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\underline{u}.\underline{u}^T = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}.(u_x\ u_y\ u_z) = \begin{pmatrix} u_x^2 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 \end{pmatrix}$$

$$\hat{u}^2 = \hat{u}.\hat{u} = \underline{u}.\underline{u}^T - I_3 = \begin{pmatrix} u_x^2-1 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2-1 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2-1 \end{pmatrix}$$

Mac Laurin

$$A(\underline{u},\theta) = I_3 + \hat{u}.s\theta + \hat{u}^2.(1-c\theta) \Rightarrow A(\underline{u},\theta) = I_3 + \hat{u}.\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right) + \hat{u}^2.\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right)$$

$$\hat{u}^3 = -\hat{u},\ \hat{u}^4 = -\hat{u}^2,\ \hat{u}^5 = \hat{u}\ ,\ \ \hat{u}^6 = \hat{u}^2$$

$$\Rightarrow \boxed{A(\underline{u},\theta) = \exp(\hat{u},\theta) = e^{\hat{u}.\theta}}$$

Basic Knowledge *Rigid body pose parameterization: orientation*

Orientation ($\underline{u}$.θ)

$$\left.\begin{aligned} A(\underline{u},\theta) &= I_3 + \hat{u}.s\theta + \hat{u}^2.(1-c\theta) \\ A(\underline{u},\theta)^T &= I_3 - \hat{u}.s\theta + \hat{u}^2.(1-c\theta) \\ A(\underline{u},\theta) - A(\underline{u},\theta)^T &= 2.\hat{u}.s\theta \end{aligned}\right\}$$

$$\boxed{[u.s\theta]^{\times} = [u.s\theta]^{\wedge} = \frac{A(\underline{u},\theta) - A(\underline{u},\theta)^T}{2}}$$

$$\left.\begin{aligned} Trace(A(\underline{u},\theta)) &= Trace(I_3 + \hat{u}.s\theta + \hat{u}^2.(1-c\theta)) \\ Trace(A(\underline{u},\theta)) &= 3 + (1-c\theta).Trace(\hat{u}^2) \\ Trace(A(\underline{u},\theta)) &= 3 + (1-c\theta).(-2) \\ Trace(A(\underline{u},\theta)) &= 1 + 2.c\theta \end{aligned}\right\}$$

$$\boxed{\begin{aligned} Tr(A(\underline{u},\theta)) &= 1 + 2.c\theta \\ \cos\theta &= \frac{Tr(A(\underline{u},\theta)) - 1}{2} \end{aligned}}$$

$$\begin{pmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{pmatrix} - \begin{pmatrix} s_x & s_y & s_z \\ n_x & n_y & n_z \\ a_x & a_y & a_z \end{pmatrix} = \begin{pmatrix} 0 & n_x - s_y & a_x - s_z \\ s_y - n_x & 0 & a_y - n_z \\ s_z - a_x & n_z - a_y & 0 \end{pmatrix}$$

$s_y - n_x = 2\, u_z\, s\theta$
$a_x - s_z = 2\, u_y\, s\theta$
$n_z - a_y = 2\, u_x\, s\theta$

$$\boxed{\sin\theta = \pm\frac{1}{2}\sqrt{(s_y - n_x)^2 + (a_x - s_z)^2 + (n_z - a_y)^2}}$$

$$\boxed{\underline{u} = \frac{1}{2.\sin\theta}\begin{pmatrix} n_z - a_y \\ a_x - s_z \\ s_y - n_x \end{pmatrix}}$$

Basic Knowledge *Rigid body pose parameterization: orientation*

Quaternion (<u>$\lambda_1,\lambda_2,\lambda_3,\lambda_4$</u>) $\theta \in [-180; 180]$

$$\lambda_1 = \cos\left(\frac{\theta}{2}\right)$$

$$\lambda_2 = u_x.\sin\left(\frac{\theta}{2}\right)$$

$$\lambda_3 = u_y.\sin\left(\frac{\theta}{2}\right)$$

$$\lambda_4 = u_z.\sin\left(\frac{\theta}{2}\right)$$

$$\lambda_1 = \frac{1}{2}.\sqrt{s_x + n_y + a_z + 1}$$

$$\lambda_2 = \frac{1}{2}.\mathrm{sign}(n_z - a_y).\sqrt{s_x - n_y - a_z + 1}$$

$$\lambda_3 = \frac{1}{2}.\mathrm{sign}(a_x - s_z).\sqrt{-s_x + n_y - a_z + 1}$$

$$\lambda_4 = \frac{1}{2}.\mathrm{sign}(s_y - n_x).\sqrt{-s_x - n_y + a_z + 1}$$

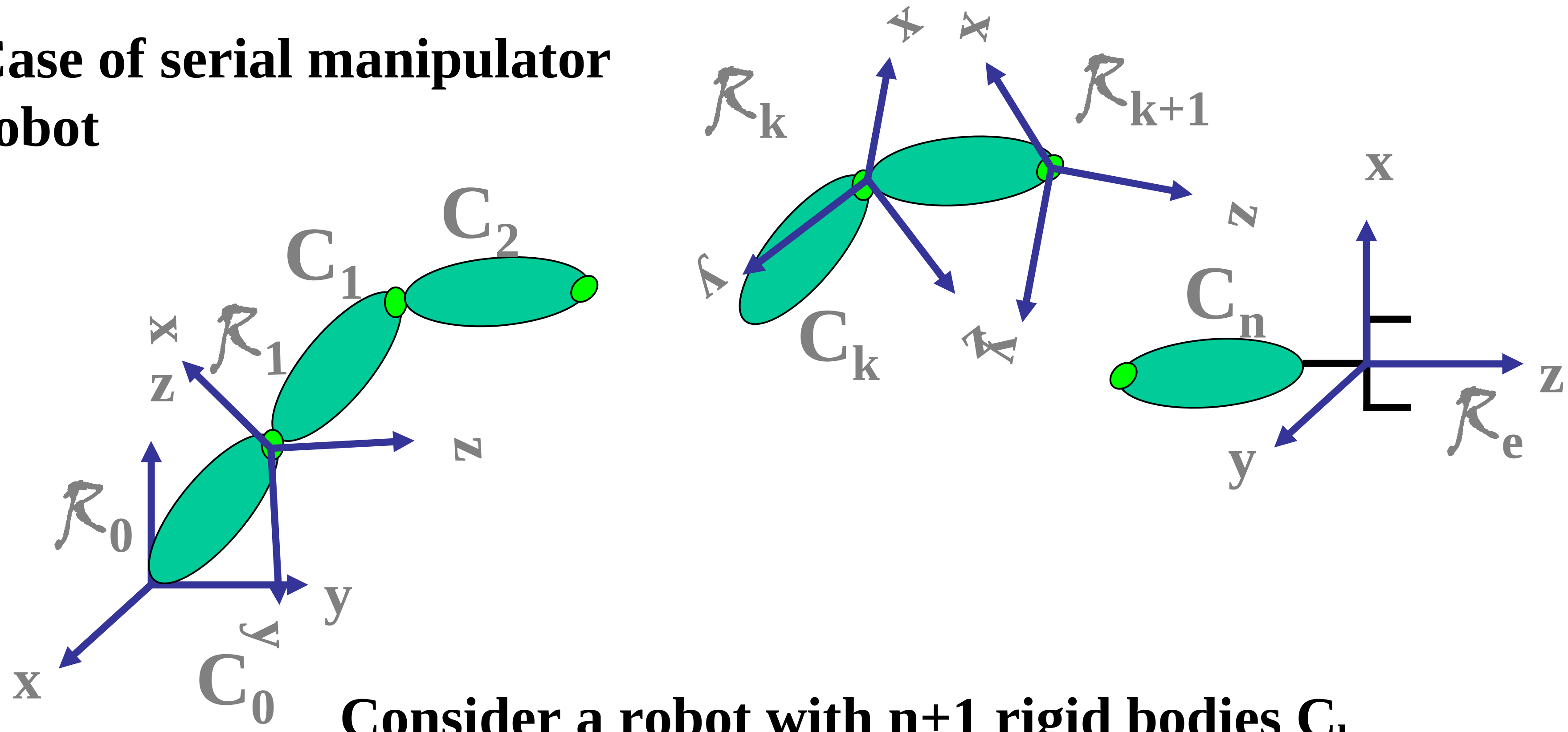
$$A_{Quat} = \begin{pmatrix} 2.(\lambda_1^2 + \lambda_2^2) - 1 & 2.(\lambda_2.\lambda_3 - \lambda_1.\lambda_4) & 2.(\lambda_2.\lambda_4 + \lambda_1.\lambda_3) \\ 2.(\lambda_2.\lambda_3 + \lambda_1.\lambda_4) & 2.(\lambda_1^2 + \lambda_3^2) - 1 & 2.(\lambda_3.\lambda_4 - \lambda_1.\lambda_2) \\ 2.(\lambda_2.\lambda_4 - \lambda_1.\lambda_3) & 2.(\lambda_3.\lambda_4 + \lambda_1\lambda_2) & 2.(\lambda_1^2 + \lambda_4^2) - 1 \end{pmatrix}$$

$$2\lambda_1^2 - 1 = \cos\theta$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$$

Basic Knowledge *Multi-Rigid bodies*

Case of serial manipulator robot



Consider a robot with n+1 rigid bodies C_k

We associate n+1 frames

C_0 is the base of the robot (fixed)

Basic Knowledge *Multi-Rigid bodies*

Case of serial manipulator robot

The problem to solve is to obtain the position and orientation of the end effector frame R_e in the fixed frame R_0

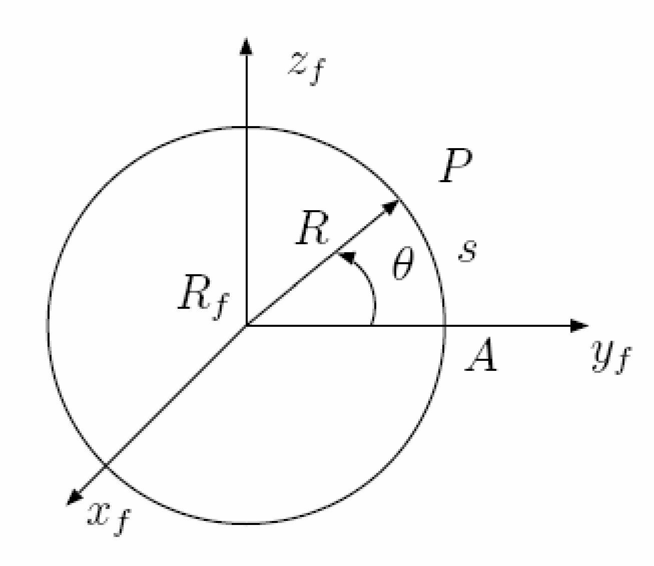
$$^{k}T_{k+1} = \begin{pmatrix} ^{k}R_{k+1} & ^{k}P_{k+1} \\ 000 & 1 \end{pmatrix}$$

Elementary frame transform

$$^{0}T_{e} = {}^{0}T_{1} \cdot {}^{1}T_{2} \cdot {}^{2}T_{3} \cdots {}^{n-1}T_{n} \cdot {}^{n}T_{e}$$

Basic Knowledge *Rigid body kinematics*

Circular motion



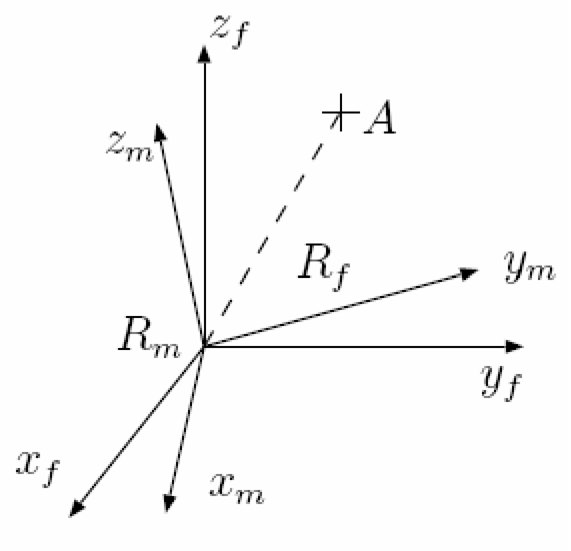
$$s = R.\theta$$

$$\boxed{\nu = \frac{ds}{dt} = R.\frac{d\theta}{dt} = R.\omega}$$

v: tangential velicity
ω: angular velocity

Basic Knowledge *Rigid body kinematics*

Rotating frame



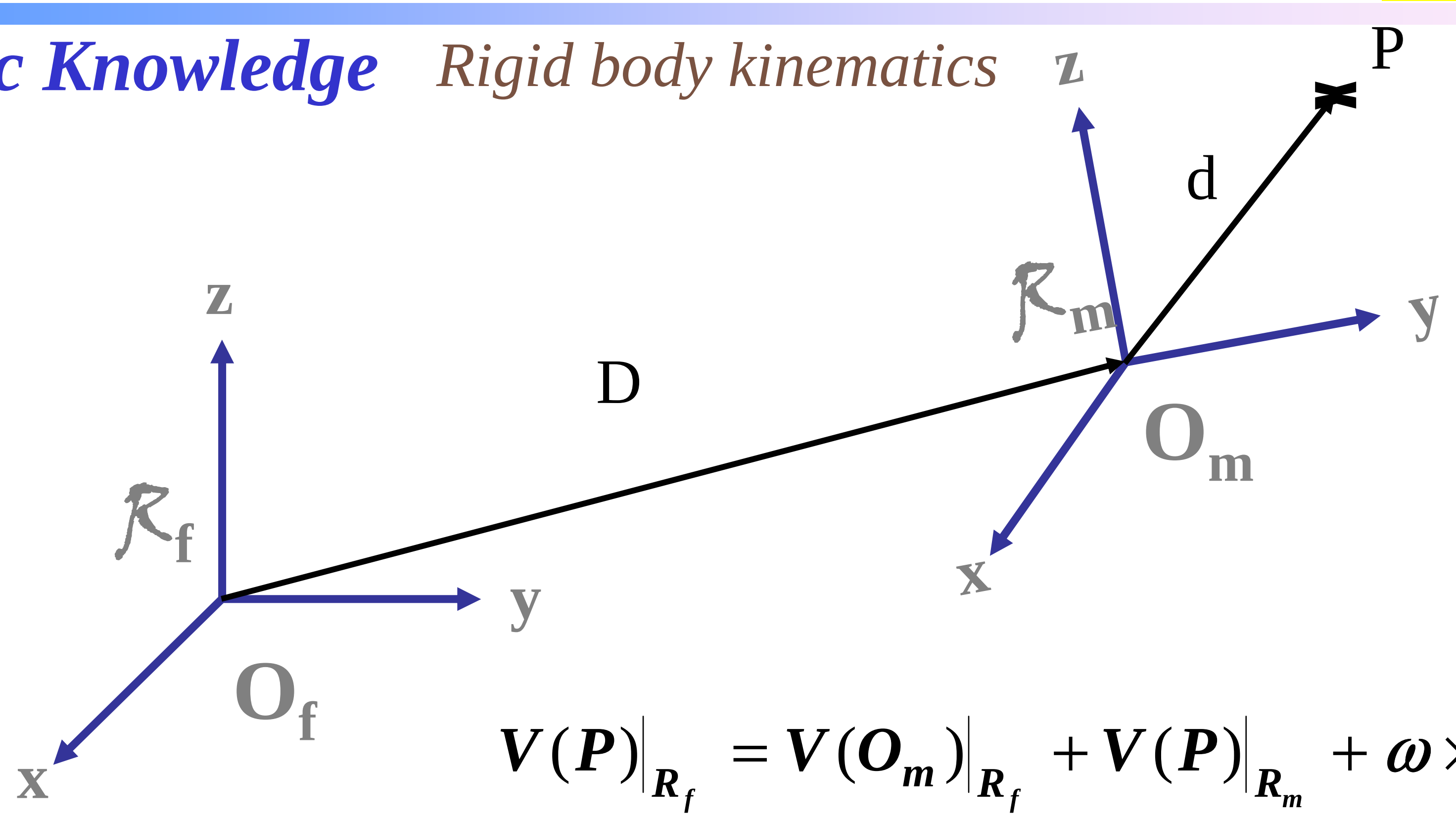
R_f: fixed frame (origin O fixed)
R_m: mobile frame just in rotation w.r.t R_f

$$\left(\frac{dA}{dt}\right)_{R_f} = \left(\frac{dA}{dt}\right)_{R_m} + \omega \wedge \underline{A}$$

$$\boxed{V(A)_{R_f} = V(A)_{R_m} + \omega \wedge \underline{A}}$$

ω: angular velocity

Basic Knowledge *Rigid body kinematics*



$$V(P)\big|_{R_f} = V(O_m)\big|_{R_f} + V(P)\big|_{R_m} + \omega \times \overline{O_m P}$$

$$\frac{d}{dt}(d)\bigg|_{R_f} = \dot{D} + \frac{d}{dt}(d)\bigg|_{R_m} + \omega \times d$$

$\mathcal{R}_f$: fixed frame
$\mathcal{R}_m$: Mobile frame
P : one point in $\mathcal{R}_m$

ω : angular velocity of $\mathcal{R}_m$ relative to $\mathcal{R}_f$

Basic Knowledge *Rigid body kinematics*

Using this relation we can established
The kinematic evolution of a multi-rigidbody robot
See next slide

Kinematic

$$\boldsymbol{V}(\boldsymbol{P})\big|_{R_f} = \boldsymbol{V}(\boldsymbol{O_m})\big|_{R_f} + \boldsymbol{V}(\boldsymbol{P})\big|_{R_m} + \boldsymbol{\omega} \times \overline{\boldsymbol{O_m P}}$$

Remarks :

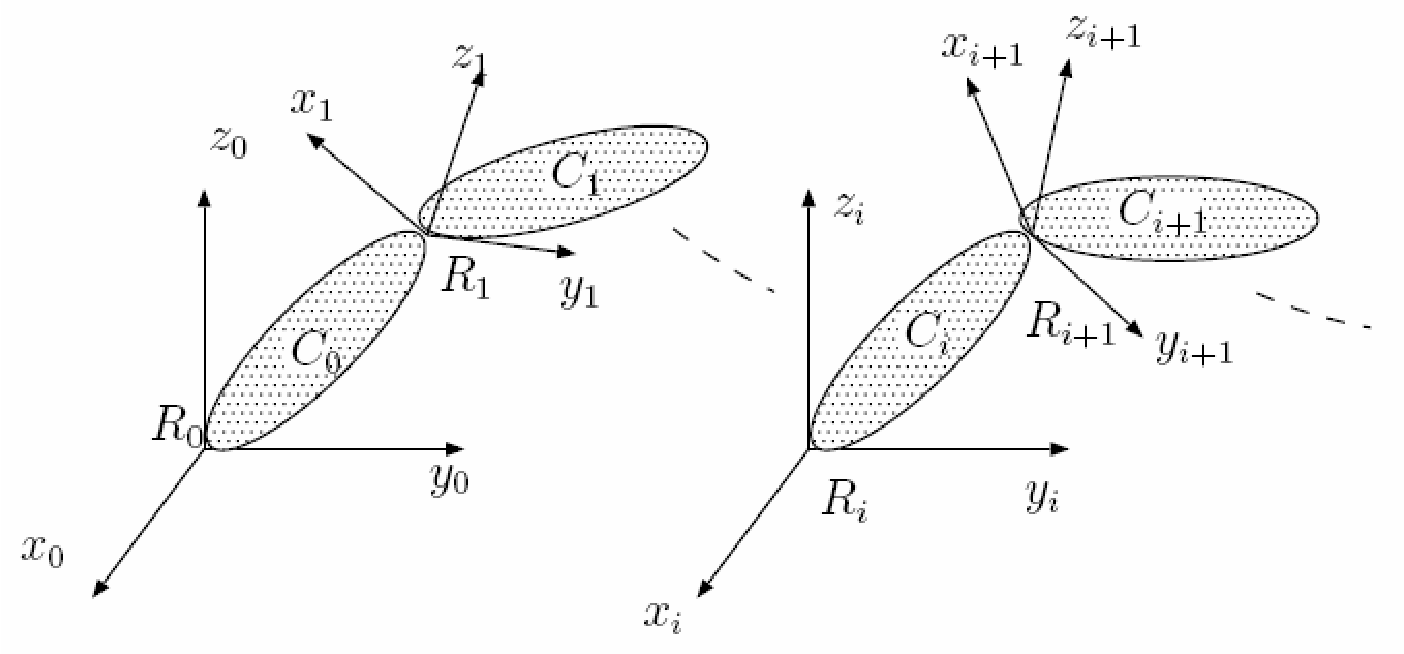
If D=0 then $\boldsymbol{V}(\boldsymbol{P})\big|_{R_f} = \boldsymbol{V}(\boldsymbol{P})\big|_{R_m} + \boldsymbol{\omega} \times \overline{\boldsymbol{O_m P}}$

If D=0 and $\boldsymbol{V}(\boldsymbol{P})\big|_{R_m} = \boldsymbol{0}$ **then**

$$\begin{aligned}\boldsymbol{V}(\boldsymbol{P})\big|_{R_f} &= \boldsymbol{\omega} \times \overline{\boldsymbol{O_m P}} \\ &= -\overline{\boldsymbol{O_m P}} \times \boldsymbol{\omega} \\ &= \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{d}\end{aligned}$$

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \boldsymbol{AS}[\omega] = [\omega]^\wedge = [\hat{\omega}]$$

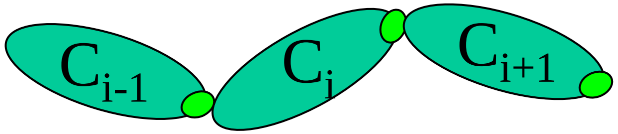
Basic Knowledge *Multi-Rigid bodies kinematics*



$$\boxed{\underbrace{V(O_{i+1})^{(R_0)}_{(R_i)}}_{\text{Velocity of } O_{i+1} \text{ w.r.t } R_0 \text{ expressed in } R_i} = \underbrace{V(O_i)^{(R_0)}_{(R_i)}}_{\text{Velocity of } O_i \text{ w.r.t } R_0 \text{ expressed in } R_i} + \underbrace{V(O_{i+1})^{(R_i)}_{(R_i)}}_{\text{Velocity of } O_{i+1} \text{ w.r.t } R_i \text{ expressed in } R_i} + \underbrace{\Omega_i{}^{(R_0)}_{(R_i)}}_{\text{Angular velocity of } R_i \text{ w.r.t } R_0 \text{ expressed in } R_i} \wedge O_iO_{i+1}}$$

Basic Knowledge *Multi-Rigid bodies kinematics*

$$\boxed{V(O_{i+1})^{(R_0)}_{(R_i)} = V(O_i)^{(R_0)}_{(R_i)} + V(O_{i+1})^{(R_i)}_{(R_i)} + \Omega_i{}^{(R_0)}_{(R_i)} \wedge O_iO_{i+1}}$$



$$\boxed{V(O_{i+1})^{(R_0)}_{(R_i)} = {}^iA_{i+1}.V(O_{i+1})^{(R_0)}_{(R_{i+1})}}$$

$$\boxed{V(O_i)^{(R_0)}_{(R_i)} = {}^iA_{i-1}.V(O_i)^{(R_0)}_{(R_{i-1})}}$$

$$\Omega_i{}^{(R_0)}_{(R_{i-1})} = \Omega_i{}^{(R_{i-1})}_{(R_{i-1})} + \Omega_{i-1}{}^{(R_0)}_{(R_{i-1})}$$

$$\Omega_i{}^{(R_0)}_{(R_i)} = {}^iA_{i-1}.\Omega_i{}^{(R_0)}_{(R_{i-1})}$$

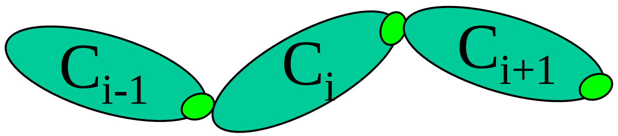
$$\boxed{\underbrace{\Omega_i{}^{(R_0)}_{(R_i)}} = \underbrace{\Omega_i{}^{(R_{i-1})}_{(R_i)}} + \underbrace{{}^iA_{i-1}.\Omega_{i-1}{}^{(R_0)}_{(R_{i-1})}}}$$

Angular velocity of R_{i-1} w.r.t R_0 expressed in R_{i-1}

Angular velocity of R_i w.r.t R_{i-1} expressed in R_i

Angular velocity of R_i w.r.t R_0 expressed in R_i

Basic Knowledge *Multi-Rigid bodies kinematics*



$$\boxed{V(O_{i+1})^{(R_0)}_{(R_i)} = V(O_i)^{(R_0)}_{(R_i)} + V(O_{i+1})^{(R_i)}_{(R_i)} + \Omega_i{}^{(R_0)}_{(R_i)} \wedge O_iO_{i+1}}$$

$$\boxed{V(O_{i+1})^{(R_0)}_{(R_i)} = {}^iA_{i+1}.V(O_{i+1})^{(R_0)}_{(R_{i+1})}}$$

$$\boxed{V(O_i)^{(R_0)}_{(R_i)} = {}^iA_{i-1}.V(O_i)^{(R_0)}_{(R_{i-1})}}$$

$$\Omega_i{}^{(R_0)}_{(R_{i-1})} = \Omega_i{}^{(R_{i-1})}_{(R_{i-1})} + \Omega_{i-1}{}^{(R_0)}_{(R_{i-1})}$$

$$\Omega_i{}^{(R_0)}_{(R_i)} = {}^iA_{i-1}.\Omega_i{}^{(R_0)}_{(R_{i-1})}$$

$$\boxed{\underbrace{\Omega_i{}^{(R_0)}_{(R_i)}} = \underbrace{\Omega_i{}^{(R_{i-1})}_{(R_i)}} + {}^iA_{i-1}.\underbrace{\Omega_{i-1}{}^{(R_0)}_{(R_{i-1})}}}$$

Angular velocity of R_{i-1} w.r.t R_0 expressed in R_{i-1}

Angular velocity of R_i w.r.t R_{i-1} expressed in R_i

Angular velocity of R_i w.r.t R_0 expressed in R_i

Basic Knowledge *Multi-Rigid bodies kinematics*

Considering two frames R_a and R_b rigidly linked (case for R_n and R_E)

$$\begin{aligned}
V(O_a)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} &= V(O_b)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} + \cancel{V(O_a)\,|_{\mathcal{R}_b}^{\mathcal{R}_b}} + \Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \wedge O_bO_a\,|_{\mathcal{R}_b} \\
&= V(O_b)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} - AS[O_bO_a].\Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
&= V(O_b)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} - AS[{}^bP_a].\Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
&= V(O_b)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} - AS[-{}^bR_a.{}^aP_b].\Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
&= V(O_b)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} + {}^bR_a.AS[{}^aP_b].{}^aR_b.\Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0}
\end{aligned}$$

$$\begin{aligned}
\Omega_a\,|_{\mathcal{R}_a}^{\mathcal{R}_0} &= \cancel{\Omega_a\,|_{\mathcal{R}_b}^{\mathcal{R}_b}} + {}^aR_b.\Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
&= {}^aR_b.\Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0}
\end{aligned}$$

$$V(O_a)\,|_{\mathcal{R}_b}^{\mathcal{R}_0} = {}^bR_a.V(O_a)\,|_{\mathcal{R}_a}^{\mathcal{R}_0} \quad\Rightarrow\quad V(O_a)\,|_{\mathcal{R}_a}^{\mathcal{R}_0} = {}^aR_b.V(O_a)\,|_{\mathcal{R}_b}^{\mathcal{R}_0}$$

$$\begin{pmatrix} V_a\,|_{\mathcal{R}_a}^{\mathcal{R}_0} \\ \Omega_a\,|_{\mathcal{R}_a}^{\mathcal{R}_0} \end{pmatrix}_{\mathcal{R}_a}^{\mathcal{R}_0} = \begin{pmatrix} {}^aR_b & AS[{}^aP_b].{}^aR_b \\ O_{3\times3} & {}^aR_b \end{pmatrix} . \begin{pmatrix} V_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \\ \Omega_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} \end{pmatrix}_{\mathcal{R}_b}^{\mathcal{R}_0}$$

twist

$$T_a\,|_{\mathcal{R}_a}^{\mathcal{R}_0} = \begin{pmatrix} {}^aR_b & AS[{}^aP_b].{}^aR_b \\ O_{3\times3} & {}^aR_b \end{pmatrix} .T_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0} = {}^a\mathcal{W}_b.T_b\,|_{\mathcal{R}_b}^{\mathcal{R}_0}$$

$$\boxed{\begin{gathered} T_a = {}^a\mathcal{W}_b.T_b \\ {}^a\mathcal{W}_b = \begin{pmatrix} {}^aR_b & AS[{}^aP_b].{}^aR_b \\ O_{3\times3} & {}^aR_b \end{pmatrix} \end{gathered}}$$

Basic Knowledge *Multi-Rigid bodies kinematics*

Considering two frames R_i and R_j and a twist $V_i=(v_i,\omega_i)^T$ expressed in O_i

We wish to compute the corresponding twist $V_j=(v_j,\omega_j)^T$ expressed in O_j

$$\omega_j = \omega_i$$
$$v_j = v_i + \omega_i \times L_{i,j}$$

$$\begin{bmatrix} v_j \\ \omega_j \end{bmatrix} = \begin{bmatrix} I_3 & -\hat{L}_{i,j} \\ 0_3 & I_3 \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

Projection

$$\begin{bmatrix} {}^i v_j \\ {}^i \omega_j \end{bmatrix} = \begin{bmatrix} I_3 & -{}^i\hat{P}_j \\ 0_3 & I_3 \end{bmatrix} \begin{bmatrix} {}^i v_i \\ {}^i \omega_i \end{bmatrix}$$

Since ${}^j v_j = {}^j R_i\, {}^i v_j$ and ${}^j\omega_j = {}^j R_i\, {}^i\omega_j$

$${}^j V_j = {}^j S_i\, {}^i V_i \quad \text{with} \quad {}^j S_i = \begin{bmatrix} {}^j R_i & -{}^j R_i\, {}^i\hat{P}_j \\ 0_3 & {}^j R_i \end{bmatrix}$$

$${}^i S_j^{-1} = \begin{bmatrix} {}^i R_j & {}^i\hat{P}_j\, {}^i R_j \\ 0_3 & {}^i R_j \end{bmatrix} = {}^j S_i$$

$${}^j S_i = \begin{bmatrix} {}^i R_j & {}^i\hat{P}_j\, {}^i R_j \\ 0_3 & {}^i R_j \end{bmatrix} = \begin{bmatrix} {}^i R_j & -{}^i R_j\, {}^j\hat{P}_i \\ 0_3 & {}^i R_j \end{bmatrix}$$

Basic Knowledge *Differential translation and rotation of frames*

Consider a differential translation vector **dP**i expressing the translation of the origin of frame Ri, and of a differential rotation vector δi, equal to **u**i.dθ, representing the rotation of an angle dθ about an axis, with unit vector **u**i, passing through the origin Oi.

$$ {}^{i}\mathbf{T}_j + d\,{}^{i}\mathbf{T}_j = \mathbf{Trans}({}^{i}dx_i, {}^{i}dy_i, {}^{i}dz_i)\,\mathbf{Rot}({}^{i}\mathbf{u}_i, d\theta)\,{}^{i}\mathbf{T}_j $$

$$ {}^{i}\mathbf{T}_j + d\,{}^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_j\,\mathbf{Trans}({}^{j}dx_j, {}^{j}dy_j, {}^{j}dz_j)\,\mathbf{Rot}({}^{j}\mathbf{u}_j, d\theta) $$

$$ d\,{}^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_j\,{}^{j}\Delta \qquad d\,{}^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_j\,[\mathbf{Trans}({}^{j}dx_j, {}^{j}dy_j, {}^{j}dz_j)\,\mathbf{Rot}({}^{j}\mathbf{u}_j, d\theta) - \mathbf{I}_4] $$

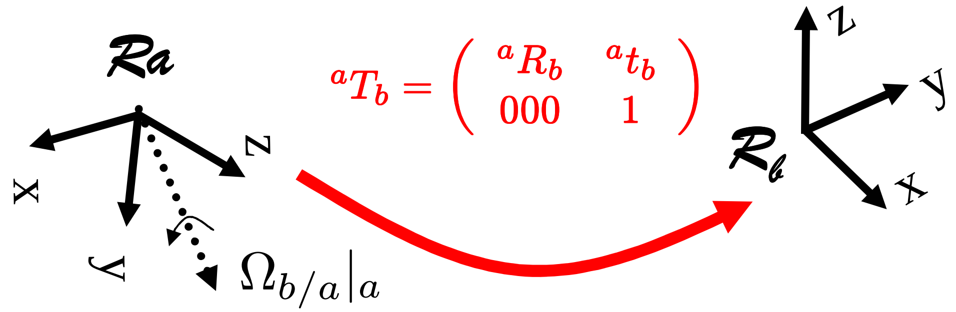
$$ d\,{}^{i}\mathbf{T}_j = {}^{i}\Delta\,{}^{i}\mathbf{T}_j \qquad d\,{}^{i}\mathbf{T}_j = [\mathbf{Trans}({}^{i}dx_i, {}^{i}dy_i, {}^{i}dz_i)\,\mathbf{Rot}({}^{i}\mathbf{u}_i, d\theta) - \mathbf{I}_4]\,{}^{i}\mathbf{T}_j $$

The differential transformation matrix Δ is defined as [Paul 81]

$$ \Delta = [\mathbf{Trans}(dx, dy, dz)\,\mathbf{Rot}(\mathbf{u}, d\theta) - \mathbf{I}_4] $$

$$ {}^{j}\Delta = \begin{bmatrix} & {}^{j}\hat{\delta}_j & & {}^{j}dP_j \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & {}^{j}\hat{\mathbf{u}}_j\,d\theta & & {}^{j}dP_j \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} {}^{j}dP_j \\ {}^{j}\delta_j \end{bmatrix} = {}^{j}S_i \begin{bmatrix} {}^{i}dP_i \\ {}^{i}\delta_i \end{bmatrix} $$

Basic Knowledge *Angular velocity properties*



$$R = {}^a R_b$$

$$R.R^T = I3$$

$$\dot{R}.R^T + R.\dot{R}^T = 0 \Rightarrow \dot{R}.R^T = -R.\dot{R}^T = -\left(\dot{R}.R^T\right)^T$$

$$\dot{R}.R^T = S(t) \Rightarrow \dot{R} = S(t).R$$

Consider $p_b(t)$ constant $\quad p_a(t) = R(t).p_b(t)$

$$\dot{p}_a(t) = \dot{R}(t).p_b(t) \Rightarrow \dot{p}_a(t) = S(t).R.p_b(t)$$

$$\left.\begin{array}{l} \dot{p}_a(t) = \Omega \times p_a(t) \\ \dot{p}_a(t) = \Omega \times (R.p_b(t)\,) \end{array}\right\} \Rightarrow \dot{p}_a(t) = [\Omega]_\times .R.p_b(t)$$

$$\Rightarrow S(t) = [\Omega]_\times$$

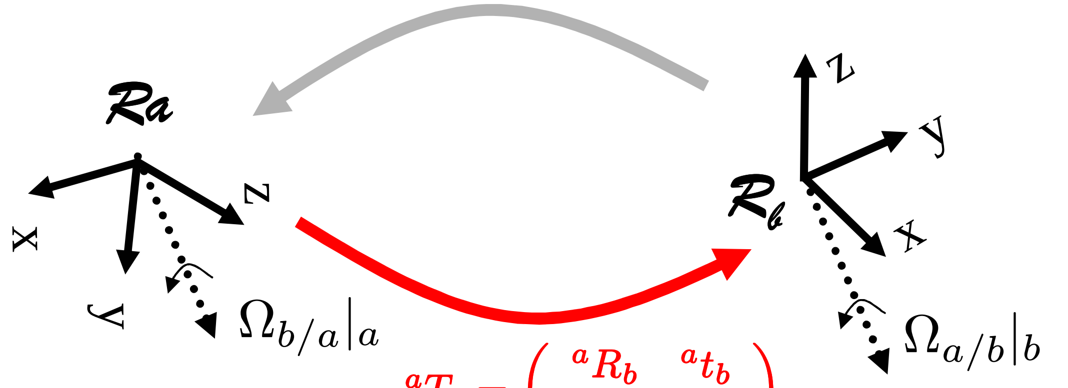
$$\left[\Omega_{b/a}|_a\right]_\times = {}^a\dot{R}_b.{}^aR_b^T$$

Basic Knowledge *Angular velocity properties*

$$ {}^bT_a = \begin{pmatrix} {}^bR_a & {}^bt_a \\ 000 & 1 \end{pmatrix} $$

${}^bR_a$ represents the orientation

${}^bt_a$ represents the position



$$ {}^aT_b = \begin{pmatrix} {}^aR_b & {}^at_b \\ 000 & 1 \end{pmatrix} $$

$\Omega_{b/a}|_a$ $\quad$ $\Omega_{b/a}|_a = -\Omega_{a/b}|_a$ $\quad$ $\Omega_{b/a}|_a = {}^aR_b.\Omega_{b/a}|_b$

$\Omega_{a/b}|_b$ $\quad$ $\Omega_{a/b}|_b = -\Omega_{b/a}|_b$ $\quad$ $\Omega_{a/b}|_b = {}^bR_a.\Omega_{a/b}|_a$

$$ \left[\Omega_{b/a}|_a\right]_\times = {}^a\dot{R}_b.{}^aR_b^T $$

$$ \left[\Omega_{a/b}|_b\right]_\times = {}^b\dot{R}_a.{}^bR_a^T $$

Basic Knowledge *Representation of forces (wrench)*

A collection of forces and moments acting on a body can be reduced to a *wrench* **F**i at point Oi, which is composed of a force **f**i at Oi and a moment **m**i about Oi:

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{m}_i \end{bmatrix}$$

Note that the vector field of the moments constitutes a screw where the vector of the screw is $\mathbf{f}_i$. Thus, the wrench forms a screw.

Consider a given wrench ${}^i\mathbf{F}_i$, expressed in frame R_i. For calculating the equivalent wrench ${}^j\mathbf{F}_j$, we use the transformation matrix between screws such that:

$$\begin{bmatrix} {}^j\mathbf{m}_j \\ {}^j\mathbf{f}_j \end{bmatrix} = {}^jS_i \begin{bmatrix} {}^i\mathbf{m}_i \\ {}^i\mathbf{f}_i \end{bmatrix} \Rightarrow \begin{aligned} {}^j\mathbf{f}_j &= {}^j\mathbf{R}_i\,{}^i\mathbf{f}_i \\ {}^j\mathbf{m}_j &= {}^j\mathbf{R}_i\,({}^i\mathbf{f}_i x\,{}^i\mathbf{P}_j + {}^i\mathbf{m}_i) \end{aligned}$$

$$\begin{bmatrix} {}^j\mathbf{f}_j \\ {}^j\mathbf{m}_j \end{bmatrix} = {}^iS_j^{\,T} \begin{bmatrix} {}^i\mathbf{f}_i \\ {}^i\mathbf{m}_i \end{bmatrix}$$