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# Mathematical Tools for Analysis and Synthesis of Mechanisms and Robots

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# Overview

- Introduction
- Parameterizations of $SE(3)$
- Quaternions - Kinematic Mapping
- Plücker Coordinates
- Serial Chains
- Varieties-Ideals
- Constraint Equations
    - Geometric Constraint Equations
    - Elimination Method
    - Linear Implicitation Algorithm (LIA)

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- Such problems are often described by ***systems of multivariate algebraic or functional equations*** and it turns out that even relatively simple kinematic problems involving multi-parameter systems lead to complicated nonlinear equations.
- Geometric insight and geometric preprocessing are often key to the solution.

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- Algebraic constraint equations yield answers to the overall behavior of a kinematic chain $\rightarrow$ **Global Kinematics**

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- **Working Mode:** Multiple configurations of the input chains.
- **Operation Mode:** Multiple output motions of a manipulator.

# Parametrizations of $\mathrm{SE}(3)$

Euclidean displacement:

$$\gamma\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{a} \tag{1}$$

$\mathbf{A}$ proper orthogonal $3 \times 3$ matrix, $\mathbf{a}, \mathbf{x} \in \mathbb{R}^3 \ldots$ vector

- group of Euclidean displacements: $\mathrm{SE}(3)$
- $\mathrm{SE}(3)$ is a non-commutative group of transformations.
- Two notations to collect rotation and translation in a *homogeneous* $4 \times 4$ transformation matrix.

$$\begin{bmatrix} w \\ \mathbf{x} \end{bmatrix} \mapsto \begin{bmatrix} 1 & \mathbf{o}^T \\ \mathbf{a} & \mathbf{A} \end{bmatrix} \cdot \begin{bmatrix} w \\ \mathbf{x} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{x} \\ w \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{o}^T & 1 \\ \mathbf{A} & \mathbf{a} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ w \end{bmatrix}. \tag{2}$$

classical European notation American notation

# Parametrizations of the rotation matrix **A**

Parametrizations are constructed from elementary properties of **A**:

"proper orthogonal": columns are orthogonal unit vectors and the determinant is 1.

**A** has 9 entries but only 3 are independent!

- *Euler angles.* Every Euclidean rotation matrix can be parameterized with three rotations about three non coplanar axes.

$$\begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

- Elementary rotations about axes of the coordinate system.
- 12 essential different sequences $\rightarrow$ 12 different parameterizations,
- $\exists$ parametrization singularities (gimbal lock)

$$\mathbf{A} = \begin{bmatrix} \cos(\gamma)\cos(\beta) & -\sin(\gamma)\cos(\alpha) + \cos(\gamma)\sin(\beta)\sin(\alpha) & \sin(\gamma)\sin(\alpha) + \cos(\gamma)\sin(\beta)\cos(\alpha) \\ \sin(\gamma)\cos(\beta) & \cos(\gamma)\cos(\alpha) + \sin(\gamma)\sin(\beta)\sin(\alpha) & -\cos(\gamma)\sin(\alpha) + \sin(\gamma)\sin(\beta)\cos(\alpha) \\ -\sin(\beta) & \cos(\beta)\sin(\alpha) & \cos(\beta)\cos(\alpha) \end{bmatrix}$$

rotation angle $\varphi$ and rotation axis $[v_1, v_2, v_3]^T$

$$\cos\varphi = \frac{1}{2}(\text{trace}(A) - 1),$$

$$v_1 : v_2 : v_3 = a_{32} - a_{23} : a_{31} - a_{13} : a_{12} - a_{21}$$

- *Rodrigues Parameters*: Every rotation matrix $\mathbf{A}$ can be computed via

$$\mathbf{A} = (\mathbf{I} - \mathbf{S})^{-1} \cdot (\mathbf{I} + \mathbf{S})$$

  where $\mathbf{S}$ is $3 \times 3$ skew symmetric and $\mathbf{I}$ is identity matrix.
- with

$$\mathbf{S} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{{b_1}^2-{b_2}^2-{b_3}^2+1}{{b_1}^2+{b_2}^2+{b_3}^2+1} & 2\,\frac{b_2\,b_1-b_3}{{b_1}^2+{b_2}^2+{b_3}^2+1} & 2\,\frac{b_3\,b_1+b_2}{{b_1}^2+{b_2}^2+{b_3}^2+1} \\ 2\,\frac{b_2\,b_1+b_3}{{b_1}^2+{b_2}^2+{b_3}^2+1} & -\frac{{b_1}^2-{b_2}^2+{b_3}^2-1}{{b_1}^2+{b_2}^2+{b_3}^2+1} & -2\,\frac{-b_2\,b_3+b_1}{{b_1}^2+{b_2}^2+{b_3}^2+1} \\ 2\,\frac{b_3\,b_1-b_2}{{b_1}^2+{b_2}^2+{b_3}^2+1} & 2\,\frac{b_2\,b_3+b_1}{{b_1}^2+{b_2}^2+{b_3}^2+1} & -\frac{{b_1}^2+{b_2}^2-{b_3}^2-1}{{b_1}^2+{b_2}^2+{b_3}^2+1} \end{bmatrix}$$

$$\tan\frac{\varphi}{2} = \sqrt{b_1^2 + b_2^2 + b_3^2} \qquad v_1 : v_2 : v_3 = b_1 : b_2 : b_3$$

- parametrization singularity $\varphi = \pi$
- $b_1$ are algebraic parameters

- Euler Parameters: $b_1 = c_1/c_0, b_2 = c_2/c_0, b_3 = c_3/c_0$

$$\mathbf{A} = \begin{bmatrix} \frac{c_0^2+c_1^2-c_2^2-c_3^2}{c_0^2+c_1^2+c_2^2+c_3^2} & -2\,\frac{c_0\,c_3-c_1\,c_2}{c_0^2+c_1^2+c_2^2+c_3^2} & 2\,\frac{c_0\,c_2+c_3\,c_1}{c_0^2+c_1^2+c_2^2+c_3^2} \\ 2\,\frac{c_0\,c_3+c_1\,c_2}{c_0^2+c_1^2+c_2^2+c_3^2} & \frac{c_0^2-c_1^2+c_2^2-c_3^2}{c_0^2+c_1^2+c_2^2+c_3^2} & -2\,\frac{c_0\,c_1-c_2\,c_3}{c_0^2+c_1^2+c_2^2+c_3^2} \\ -2\,\frac{c_0\,c_2-c_3\,c_1}{c_0^2+c_1^2+c_2^2+c_3^2} & 2\,\frac{c_0\,c_1+c_2\,c_3}{c_0^2+c_1^2+c_2^2+c_3^2} & \frac{c_0^2-c_1^2-c_2^2+c_3^2}{c_0^2+c_1^2+c_2^2+c_3^2} \end{bmatrix}$$

- $c_i$ four homogeneous parameters, singularity free,
- possible normalizations $c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1$ or $c_0 = 1$
- Euler parameters are identical to the the (Hamiltonian) quaternions describing rotations

# Quaternions

The set of quaternions $\mathbb{H}$ is the vector space $\mathbb{R}^4$ together with the quaternion multiplication

$$\begin{aligned}(a_0, a_1, a_2, a_3) \star (b_0, b_1, b_2, b_3) = (&a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3, \\ &a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2, \\ &a_0 b_2 - a_1 b_3 + a_2 b_0 - a_3 b_1, \\ &a_0 b_3 - a_1 b_2 - a_2 b_1 + a_3 b_0).\end{aligned} \tag{3}$$

- The triple $(\mathbb{H}, +, \star)$ (with component wise addition) forms a skew field.
- The real numbers can be embedded into this field via $x \mapsto (x, 0, 0, 0)$
- vectors $\mathbf{x} \in \mathbb{R}^3$ are identified with quaternions of the shape $(0, \mathbf{x})$.

Every quaternion is a unique linear combination of the four basis quaternions $\mathbf{1} = (1, 0, 0, 0)$, $\mathbf{i} = (0, 1, 0, 0)$, $\mathbf{j} = (0, 0, 1, 0)$, and $\mathbf{k} = (0, 0, 0, 1)$.

The multiplication table is

| $\star$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ |
|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ |
| $\mathbf{i}$ | $\mathbf{i}$ | $-\mathbf{1}$ | $\mathbf{k}$ | $-\mathbf{j}$ |
| $\mathbf{j}$ | $\mathbf{j}$ | $-\mathbf{k}$ | $-\mathbf{1}$ | $\mathbf{i}$ |
| $\mathbf{k}$ | $\mathbf{k}$ | $\mathbf{j}$ | $-\mathbf{i}$ | $-\mathbf{1}$ |

**Example:** elementary rotations about coordinate axes:

$$r_x = \mathbf{1} + u\mathbf{i}, \qquad r_y = \mathbf{1} + v\mathbf{j}, \qquad r_z = \mathbf{1} + w\mathbf{k},$$

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*Conjugate quaternion* and *norm* are defined as

$$\overline{A} = (a_0, -a_1, -a_2, -a_3), \quad \|A\| = \sqrt{A \star \overline{A}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}. \tag{4}$$

# Kinematic mapping

Study's kinematic mapping $\varkappa$:

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pre-image of $\mathbf{x}$ is the displacement $\alpha$

$$\frac{1}{\Delta}\begin{bmatrix} \Delta & 0 & 0 & 0 \\ p & x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ q & 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ r & 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{bmatrix} \tag{5}$$

$$\begin{aligned} p &= 2(-x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2), \\ q &= 2(-x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1), \\ r &= 2(-x_0y_3 - x_1y_2 + x_2y_1 + x_3y_0), \end{aligned} \tag{6}$$

$\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2.$

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$\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2.$

$$S_6^2 : \quad x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \qquad x_i \quad \text{not all} \quad 0$$

$[x_0 : \cdots : y_3]^T$ *Study parameters* = parametrization of $SE(3)$ with dual quaternions

## $S_6^2$ : Study quadric

Named after

Eduard Study (23.3.1862-6.1.1930)

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How do we get the Study parameters when a proper orthogonal matrix $\mathbf{A} = [a_{ij}]$ and the translation vector $\mathbf{a} = [a_k]^T$ are given?

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Rotation part:

$$\begin{aligned} x_0 : x_1 : x_2 : x_3 &= 1 + a_{11} + a_{22} + a_{33} : a_{32} - a_{23} : a_{13} - a_{31} : a_{21} - a_{12} \\ &= a_{32} - a_{23} : 1 + a_{11} - a_{22} - a_{33} : a_{12} + a_{21} : a_{31} + a_{13} \\ &= a_{13} - a_{31} : a_{12} + a_{21} : 1 - a_{11} + a_{22} - a_{33} : a_{23} + a_{32} \\ &= a_{21} - a_{12} : a_{31} + a_{13} : a_{23} - a_{32} : 1 - a_{11} - a_{22} + a_{33} \end{aligned} \tag{7}$$

In general, all four proportions of Eq. (7) yield the same result. Translation part:

$$\begin{aligned} 2y_0 &= \phantom{-}a_1x_1 + a_2x_2 + a_3x_3, \quad & 2y_1 &= -a_1x_0 + a_3x_2 - a_2x_3, \\ 2y_2 &= -a_2x_0 - a_3x_1 + a_1x_3, \quad & 2y_3 &= -a_3x_0 + a_2x_1 - a_1x_2. \end{aligned} \tag{8}$$

# Example 1

Rotation about x-axis:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{bmatrix}.$$

Its kinematic image, computed via (7) and (8) is

$$\mathbf{r} = [1 + \cos\varphi : \sin\varphi : 0 : 0 : 0 : 0 : 0 : 0].$$

As $\varphi$ varies in $[0, 2\pi)$, $\mathbf{r}$ describes a straight line on the Study quadric which reads after algebraization with half-tangent substitution

$$\mathbf{r}_x = [1 : u : 0 : 0 : 0 : 0 : 0 : 0].$$

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The other two elementary rotations about $y-$ and $z-$axis can be written in Study parameters as:

$$\mathbf{r}_y = [1 : 0 : v : 0 : 0 : 0 : 0 : 0], \mathbf{r}_z = [1 : 0 : 0 : w : 0 : 0 : 0 : 0].$$

# Example 2

R-P-R chain (P fixed translation)

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2a & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(s) & 0 & -\sin(s) \\ 0 & 0 & 1 & 0 \\ 0 & \cos(s) & 0 & \sin(s) \end{bmatrix},$$

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$$\mathbf{L} = \mathbf{M} \cdot \mathbf{N} \cdot \mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(s) & 0 & -\sin(s) \\ -2\sin(t)\,a & -\sin(t)\cos(s) & \cos(t) & -\sin(t)\sin(s) \\ 2\cos(t)\,a & \cos(t)\cos(s) & \sin(t) & \cos(t)\sin(s) \end{bmatrix}.$$

Its kinematic image, computed via (7) and (8) is

$$\mathbf{l} = \begin{bmatrix} 1+\cos(s)+\cos(t)+\cos(t)\sin(s) \\ \sin(t)+\sin(t)\sin(s) \\ -\sin(s)-\cos(t)\cos(s) \\ -\sin(t)\cos(s) \\ -\sin(t)\,a\,(-\sin(s)-\cos(t)\cos(s))-\cos(t)\,a\sin(t)\cos(s) \\ \cos(t)\,a\,(-\sin(s)-\cos(t)\cos(s))-(\sin(t))^2\,a\cos(s) \\ \sin(t)\,a\,(1+\cos(s)+\cos(t)+\cos(t)\sin(s))-\cos(t)\,a\,(\sin(t)+\sin(t)\sin(s)) \\ -\cos(t)\,a\,(1+\cos(s)+\cos(t)+\cos(t)\sin(s))-\sin(t)\,a\,(\sin(t)+\sin(t)\sin(s)) \end{bmatrix}.$$

Its kinematic image, computed via (7) and (8) is

$$\mathbf{I} = \begin{bmatrix} 1 + \cos(s) + \cos(t) + \cos(t)\sin(s) \\ \sin(t) + \sin(t)\sin(s) \\ -\sin(s) - \cos(t)\cos(s) \\ -\sin(t)\cos(s) \\ -\sin(t)\,a\,(-\sin(s) - \cos(t)\cos(s)) - \cos(t)\,a\sin(t)\cos(s) \\ \cos(t)\,a\,(-\sin(s) - \cos(t)\cos(s)) - (\sin(t))^2\,a\cos(s) \\ \sin(t)\,a\,(1 + \cos(s) + \cos(t) + \cos(t)\sin(s)) - \cos(t)\,a\,(\sin(t) + \sin(t)\sin(s)) \\ -\cos(t)\,a\,(1 + \cos(s) + \cos(t) + \cos(t)\sin(s)) - \sin(t)\,a\,(\sin(t) + \sin(t)\sin(s)) \end{bmatrix}.$$

after algebraization with half-tangent substitution:

$$\mathbf{I} = [1 : u : v : uv : -uav : av : ua : -a].$$

Planar displacements: $x_2 = x_3 = 0, y_0 = y_1 = 0$

$$\frac{1}{x_0^2 + x_3^2} \begin{bmatrix} x_0^2 + x_3^2 & 0 & 0 \\ -2(x_0 y_1 - x_3 y_2) & x_0^2 - x_3^2 & -2x_0 x_3 \\ -2(x_0 y_2 + x_3 y_1) & 2x_0 x_3 & x_0^2 - x_3^2 \end{bmatrix}$$

$\mathrm{SE}(2)$ (we omit the last row and the last column)

Spherical displacements: $y_i = 0$ ($\rightarrow$ Euler parameters!)

$$\frac{1}{\Delta} \begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1 x_2 - x_0 x_3) & 2(x_1 x_3 + x_0 x_2) \\ 2(x_1 x_2 + x_0 x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2 x_3 - x_0 x_1) \\ 2(x_1 x_3 - x_0 x_2) & 2(x_2 x_3 + x_0 x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{bmatrix} \tag{9}$$

where $\Delta = x_0^2 + x_1^2 + x_2^2 + x_3^2$. $\rightarrow$ $SO^+(3)$

generate 3-spaces on $S_6^2$

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Spherical displacements: $y_i = 0$ ($\rightarrow$ Euler parameters!)

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generate 3-spaces on $S_6^2$

more properties:

J. Selig, Geometric Fundamentals of Robotics, 2nd. ed. Springer 2005
Husty, Pfurner, Schröcker, Brunnthaler. Algebraic methods in mechanism analysis and synthesis. Robotica, 25(6):661-675, 2007.

(Hamiltonian) Quaternions are closely related to spherical kinematic mapping.

Consider a vector $\mathbf{a} = [a_1, a_2, a_3]^T$ and a matrix $\mathbf{X}$ of the shape (9).

The product $\mathbf{b} = \mathbf{X} \cdot \mathbf{a}$ can also be written as

$$B = X \star A \star \overline{X}$$

where $X = (x_0, x_1, x_2, x_3)$, $\|X\| = 1$ and $A = (0, \mathbf{a}),\ B = (0, \mathbf{b})$.

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From this follows:

*Spherical displacements* can also be described by *unit quaternions* and *spherical kinematic mapping* maps a spherical displacement to the corresponding *unit quaternion*.

General Euclidean displacements $\rightarrow$ extend the concept of quaternions.

A *dual quaternion* $Q$ is a quaternion over the ring of dual numbers

$$Q = Q_0 + \varepsilon Q_1,$$

where $\varepsilon^2 = 0$, $Q_0, Q_1$ are Hamiltonian quaternions, e.g. $Q_0 = (q_0, q_1, q_2, q_3)$.

The algebra of dual quaternions has eight basis elements $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\varepsilon$, $\varepsilon\mathbf{i}$, $\varepsilon\mathbf{j}$, and $\varepsilon\mathbf{k}$ and the multiplication table

| $\star$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ | $\varepsilon$ | $\varepsilon\mathbf{i}$ | $\varepsilon\mathbf{j}$ | $\varepsilon\mathbf{k}$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{i}$ | $\mathbf{j}$ | $\mathbf{k}$ | $\varepsilon$ | $\varepsilon\mathbf{i}$ | $\varepsilon\mathbf{j}$ | $\varepsilon\mathbf{k}$ |
| $\mathbf{i}$ | $\mathbf{i}$ | $-\mathbf{1}$ | $\mathbf{k}$ | $-\mathbf{j}$ | $\varepsilon\mathbf{i}$ | $-\varepsilon\mathbf{1}$ | $\varepsilon\mathbf{k}$ | $-\varepsilon\mathbf{j}$ |
| $\mathbf{j}$ | $\mathbf{j}$ | $-\mathbf{k}$ | $-\mathbf{1}$ | $\mathbf{i}$ | $\varepsilon\mathbf{j}$ | $-\varepsilon\mathbf{k}$ | $-\varepsilon\mathbf{1}$ | $\varepsilon\mathbf{i}$ |
| $\mathbf{k}$ | $\mathbf{k}$ | $\mathbf{j}$ | $-\mathbf{i}$ | $-\mathbf{1}$ | $\varepsilon\mathbf{k}$ | $\varepsilon\mathbf{j}$ | $-\varepsilon\mathbf{i}$ | $-\varepsilon\mathbf{1}$ |
| $\varepsilon\mathbf{1}$ | $\varepsilon$ | $\varepsilon\mathbf{i}$ | $\varepsilon\mathbf{j}$ | $\varepsilon\mathbf{k}$ | 0 | 0 | 0 | 0 |
| $\varepsilon\mathbf{i}$ | $\varepsilon\mathbf{i}$ | $-\varepsilon\mathbf{1}$ | $\varepsilon\mathbf{k}$ | $-\varepsilon\mathbf{j}$ | 0 | 0 | 0 | 0 |
| $\varepsilon\mathbf{j}$ | $\varepsilon\mathbf{j}$ | $-\varepsilon\mathbf{k}$ | $-\varepsilon\mathbf{1}$ | $\varepsilon\mathbf{i}$ | 0 | 0 | 0 | 0 |
| $\varepsilon\mathbf{k}$ | $\varepsilon\mathbf{k}$ | $\varepsilon\mathbf{j}$ | $-\varepsilon\mathbf{i}$ | $-\varepsilon\mathbf{1}$ | 0 | 0 | 0 | 0 |

Dual quaternions know two types of conjugation.

The *conjugate quaternion* and the *conjugate dual quaternion* of a dual quaternion $Q = x_0 + \varepsilon y_0 + \mathbf{x} + \varepsilon \mathbf{y}$ are defined as

$$\overline{Q} = x_0 + \varepsilon y_0 - \mathbf{x} - \varepsilon \mathbf{y} \quad \text{and} \quad Q_e = x_0 - \varepsilon y_0 + \mathbf{x} - \varepsilon \mathbf{y},$$

respectively. The norm of a dual quaternion is

$$\|Q\| = \sqrt{Q\overline{Q}}.$$

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The equation $\mathbf{b} = \mathbf{X} \cdot \mathbf{a}$ where $\mathbf{X}$ is a matrix of the shape (5) can be written as

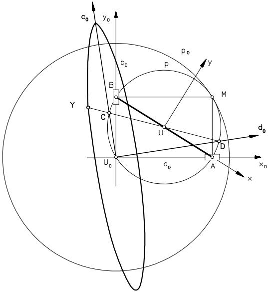
$$B = X_e \star A \star \overline{X}$$

where $X = \mathbf{x} + \varepsilon \mathbf{y}$, $\|X\| = 1$, $\mathbf{x} = (x_0, \ldots, x_3)^T$, $\mathbf{y} = (y_0, \ldots, y_3)^T$, and $\mathbf{x} \cdot \mathbf{y} = 0$.

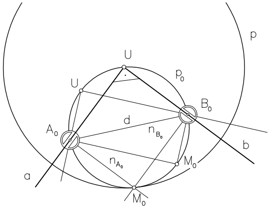
Last condition is precisely the Study condition

$A$ and $B$ are dual quaternions of the type: $A = 1 + \varepsilon \mathbf{a}, \; B = 1 + \varepsilon \mathbf{b}$

# Example: Elliptic Motion( double slider) - Oldham motion




Double slider (tramel motion)




Oldham motion

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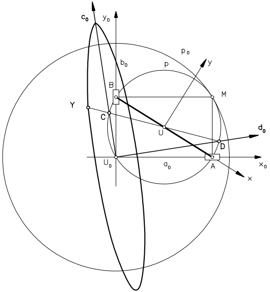
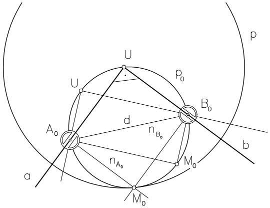



Double slider (tramel motion)




Oldham motion

$$\textit{Doubleslider}: \quad [4+(t^2+1),0,0,4t(t^2+1),0,d(3t^2-1),-dt(t^3-3,0]^T$$

$$\textit{Oldham}: \quad [4+(t^2+1),0,0,-4t(t^2+1),0,-d(3t^2-1),dt(t^3-3,0]^T$$

# Example: Elliptic Motion( double slider) - Oldham motion

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{d}{2}\frac{1-t^2}{t^2+1} & -\frac{1-t^2}{t^2+1} & \frac{2t}{t^2+1} & 0 \\ -\frac{d}{2}\frac{2t}{t^2+1} & \frac{2t}{t^2+1} & \frac{1-t^2}{t^2+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{O} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{d}{2}\frac{t^4-6t^2+1}{t^2+1} & \frac{1-t^2}{t^2+1} & -\frac{2t}{t^2+1} & 0 \\ -\frac{d}{2}\frac{2t(1-t^2)}{t^2+1} & \frac{2t}{t^2+1} & \frac{1-t^2}{t^2+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Plücker Coordinates

**Definition**

Let $X(x_0 : x_1 : x_2 : x_3)$ and $Y(y_0 : y_1 : y_2 : y_3)$ be two different points of a line $p \in P_3$, then

$$p_{ik} := \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} \quad (i, k : 0, \ldots, 3, i \neq k) \tag{10}$$

are called homogeneous Plücker-Coordinates (line coordinates) von $p$.

Out of the 12 determinants only 6 are relevant

$$\begin{aligned} p_{01} = p_1; \quad p_{02} = p_2; \quad p_{03} = p_3; \\ p_{23} = p_4; \quad p_{31} = p_5; \quad p_{12} = p_6 \end{aligned} \tag{11}$$

$$\Omega(p) := p_1 p_4 + p_2 p_5 + p_3 p_6 = \sum_{\nu=1}^{3} p_\nu p_{\nu+3} = 0 \, , \tag{12}$$

sometime also written

$$\Omega(p) = p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0 \tag{13}$$

1. The Plücker coordinates are independent of the choice of the points on the line
2. The Plücker coordinates can be interpreted as points in a five dimensional projective space $P^5$
3. $\Omega$ is a hyper quadric in $P^5$, called Plücker quadric

Plücker coordinates transform :

$$\mathbf{p} \rightarrow \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{a}^\times \mathbf{A} & \mathbf{A} \end{pmatrix} \mathbf{p}$$

$\mathbf{a}^\times$ skew symmetric matrix belonging to translation vector $\mathbf{a}$.

# Axis Coordinates

Coordinates of a plane:
$\mathbf{e} \ \ldots \ u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0 \rightarrow [u_0 : u_1 : u_2 : u_3]$.

**Definition**

Let $\mathbf{e}_1$ $[u_0 : u_1 : u_2 : u_3]$ and $\mathbf{e}_2[v_0 : v_1 : v_2 : v_3]$ be two different planes passing through the line $p$, then

$$\widehat{p}_{ik} := \begin{vmatrix} u_i & u_k \\ v_i & v_k \end{vmatrix} \ (i, k : 0, \ldots, 3; i \neq k) \tag{14}$$

are called homogeneous axis coordinates of $p$

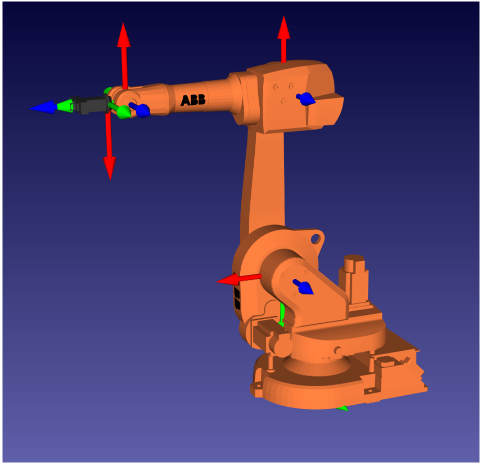
line objects:

1. A linear equation in Plücker coordinates

$$a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 + a_5p_5 + a_6p_6 = 0$$

   determines a *linear line complex* (three parametric set of lines)
2. Two linear equations in Plücker coordinates determine a *linear congruence* of lines (two parametric set of lines)
3. Three linear equations in Plücker coordinates determine a *hyperboloid* (one parametric set of lines)
4. degenerate cases exist: singular line congruence, parabolic congruence, pencils of lines, bundles of lines

# Serial robots



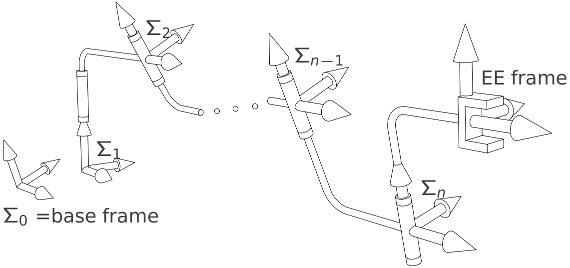



**Figure:** Coordinate frames attached to a general nR-mechanism

Forward Kinematics

$$\mathbf{D} = \mathbf{B} \cdot \mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \cdots \cdot \mathbf{M}_{n-1} \cdot \mathbf{G}_{n-1} \cdot \mathbf{M}_n \cdot \mathbf{G}_n, \tag{15}$$

where **B** is the coordinate transformation $\Sigma_0 \to \Sigma_1$, Coordinate transformation matrices

$$\mathbf{G}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_i & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ d_i & 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}$$

Rotation Matrices

$$\mathbf{M}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(u_i) & -\sin(u_i) & 0 \\ 0 & \sin(u_i) & \cos(u_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $i = 1, \ldots, n$

$a_i, d_i, \alpha_i$ . . . Denavit-Hartenberg parameters

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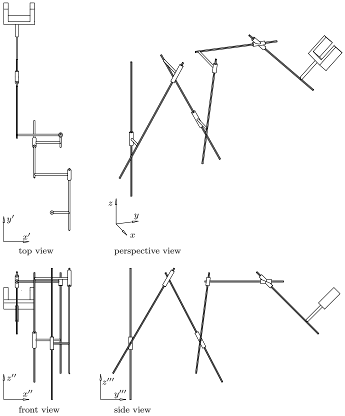
for $i = 1, \ldots, n$
$a_i, d_i, \alpha_i \ldots$ Denavit-Hartenberg parameters

The DH parameters completely determine the design of the manipulator. For an nR manipulator there are exactly $3n - 4$ DH parameters.

# Home Pose




Every serial manipulator can be brought into a pose where all axes are parallel to a plane (here yz-plane).

# Singularities without differentiation

In the columns if the Jacobian Matrix $\mathbf{J}$ are the Plücker coordinates of the instantaneous locations of the revolute axes of the robot.

In local coordinate system the axes are $\mathbf{p_i} = [0, 0, 1, 0, 0, 0]$

$$\mathbf{p_1} = [0, 0, 1, 0, 0, 0]$$

$$\mathbf{p_2} = \mathbf{A_2 p_1} \qquad \vdots$$

$\mathbf{A} = \mathbf{M}_1 \mathbf{G}_1$ written as line transform matrix

# Constraint varieties of 3R-chains

Algorithm:

1. Determine the constraint variety of a canonical serial 2R-chain
2. Add one more rotation -> algebraic representation of a canonical 3R chain
3. Add a (linear) base transformation in the image space -> general 3R chain

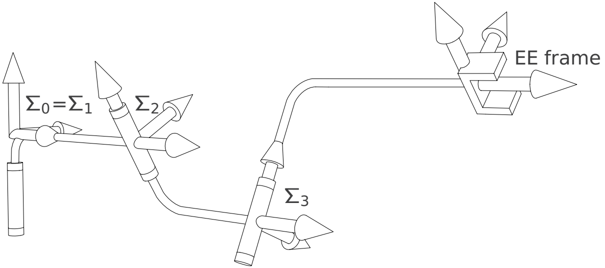



**Figure:** Canonical 3R-manipulator

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$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n : f_i(a_1, \dots, a_n) = 0, \quad \text{for all} \quad 1 \leq i \leq s\}$$

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- An ideal $I$ is a subset of $k[x_0, \dots x_n]$ that satisfies the following properties:

$$\begin{aligned} &(i)\ 0 \in I. \\ &(ii)\ \text{If } f, g \in I, \quad \text{then} \quad f + g \in I. \\ &(iii)\ \text{If } f \in I,\ g \in k \quad \text{then} \quad fg \in I. \end{aligned}$$

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D. A. Cox, J. B. Little, and D. O'Shea, Ideals, Varieties and Algorithms, Springer, third ed., 2007.

Step 1: Fix $u_1$

$$\mathbf{D} = \mathbf{F} \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \mathbf{M}_3 \cdot \mathbf{G}_3.$$

where $\mathbf{F}$ is a fixed transformation, given by $\mathbf{M}_1(u_{10}) \cdot \mathbf{G}_1$. $\mathbf{F}$ and $\mathbf{G}_3$ are coordinate transformations in the base and moving frame of the 2R-manipulator

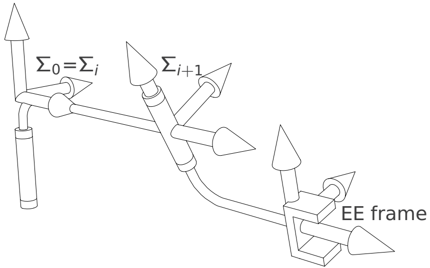


**Figure:** Canonical 2R-mechanism

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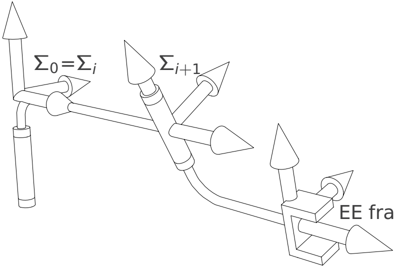


**Figure:** Canonical 2R-mechanism

matrix representation of this 2R-chain becomes

$$\mathbf{D} = \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \mathbf{M}_3.$$

## Parametric representation of the constraint variety

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} (\cos(u_2)\cos(u_3) - \sin(u_2)\sin(u_3) + 1)(1 + \cos(\alpha_2)) \\ (\cos(u_2) + \cos(u_3))\sin(\alpha_2) \\ (\sin(u_2) - \sin(u_3))\sin(\alpha_2) \\ (\cos(u_2)\sin(u_3) + \sin(u_2)\cos(u_3))(1 + \cos(\alpha_2)) \\ \frac{1}{2}a_2(\cos(u_2)\cos(u_3) - \sin(u_2)\sin(u_3) + 1)(\sin\alpha_2) \\ -\frac{1}{2}a_2(\cos(u_2) + \cos(u_3))(1 + \cos(\alpha_2)) \\ -\frac{1}{2}a_2(\sin(u_2) - \sin(u_3))(1 + \cos(\alpha_2)) \\ \frac{1}{2}a_2(\cos(u_2)\sin(u_3) + \sin(u_2)\cos(u_3))(\sin(\alpha_2)) \end{pmatrix}.$$

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By inspection and direct substitution one can verify easily that these coordinates satisfy four independent linear equations:

$$\begin{aligned}
\overline{Hc}_{11}: &\quad a_2\sin(\alpha_2)x_0 - 2(1+\cos(\alpha_2))y_0 = 0, \\
\overline{Hc}_{12}: &\quad a_2(1+\cos(\alpha_2))x_1 + 2\sin(\alpha_2)y_1 = 0, \\
\overline{Hc}_{13}: &\quad a_2(1+\cos(\alpha_2))x_2 + 2\sin(\alpha_2)y_2 = 0, \\
\overline{Hc}_{14}: &\quad a_2\sin(\alpha_2)x_3 - 2(1+\cos(\alpha_2))y_3 = 0.
\end{aligned}$$

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$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} (\cos(u_2)\cos(u_3) - \sin(u_2)\sin(u_3) + 1)(1+\cos(\alpha_2)) \\ (\cos(u_2) + \cos(u_3))\sin(\alpha_2) \\ (\sin(u_2) - \sin(u_3))\sin(\alpha_2) \\ (\cos(u_2)\sin(u_3) + \sin(u_2)\cos(u_3))(1+\cos(\alpha_2)) \\ \frac{1}{2}a_2(\cos(u_2)\cos(u_3) - \sin(u_2)\sin(u_3) + 1)(\sin\alpha_2) \\ -\frac{1}{2}a_2(\cos(u_2) + \cos(u_3))(1+\cos(\alpha_2)) \\ -\frac{1}{2}a_2(\sin(u_2) - \sin(u_3))(1+\cos(\alpha_2)) \\ \frac{1}{2}a_2(\cos(u_2)\sin(u_3) + \sin(u_2)\cos(u_3))(\sin(\alpha_2)) \end{pmatrix}.$$

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Applying half tangent substitution ($al_2 = \tan\frac{\alpha_2}{2}$) these equations rewrite to

$$\begin{aligned} \overline{Hc}_{11}: &\quad 2a_2al_2x_0 - 4y_0 = 0, \\ \overline{Hc}_{12}: &\quad 2a_2x_1 + 4al_2y_1 = 0, \\ \overline{Hc}_{13}: &\quad 2a_2x_2 + 4al_2y_2 = 0, \\ \overline{Hc}_{14}: &\quad 2a_2al_2x_3 - 4y_3 = 0. \end{aligned} \tag{16}$$

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The constraint variety of a canonical A 2-R chain is represented by four linear equations.

Step 2: Add variation of $u_1$

$$
\begin{aligned}
&Hc_1(v_1): \\
&(a_2al_2 - v_1d_2 - al_3a_1 - al_3a_3 - al_1a_1 - a_2al_2al_3al_1 - al_3v_1d_2al_1 \\
&- al_3d_3al_1v_1 - a_3al_1 - d_3v_1)x_0 + (-al_3v_1d_2 + a_2al_2al_3 + a_2al_2al_1 \\
&+ a_1 + a_3 - al_3al_1a_1 + v_1d_2al_1 - al_3a_3al_1 + d_3al_1v_1 - al_3d_3v_1)x_1 \\
&+ (a_1v_1 - d_2al_1 + al_3d_3 - d_3al_1 + a_2al_2al_1v_1 + al_3d_2 - al_3al_1a_1v_1 \\
&- al_3a_3al_1v_1 + a_3v_1 + a_2al_2al_3v_1)x_2 + (-a_3al_1v_1 + d_2 + d_3 - al_1a_1v_1 \\
&+ a_2al_2v_1 - al_3a_1v_1 + al_3d_2al_1 + al_3d_3al_1 - a_2al_2al_3al_1v_1 - al_3a_3v_1)x_3 \\
&+ 2(al_3al_1 - 1)y_0 - 2(al_3 + al_1)y_1 - 2(al_1v_1 + al_3v_1)y_2 + 2(al_3al_1v_1 - v_1)y_3 = 0
\end{aligned}
$$

Step 3: if necessary add a base transformation -> general 3R-chain

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&- al_3d_3al_1v_1 - a_3al_1 - d_3v_1)x_0 + (-al_3v_1d_2 + a_2al_2al_3 + a_2al_2al_1\\
&+ a_1 + a_3 - al_3al_1a_1 + v_1d_2al_1 - al_3a_3al_1 + d_3al_1v_1 - al_3d_3v_1)x_1\\
&+ (a_1v_1 - d_2al_1 + al_3d_3 - d_3al_1 + a_2al_2al_1v_1 + al_3d_2 - al_3al_1a_1v_1\\
&- al_3a_3al_1v_1 + a_3v_1 + a_2al_2al_3v_1)x_2 + (-a_3al_1v_1 + d_2 + d_3 - al_1a_1v_1\\
&+ a_2al_2v_1 - al_3a_1v_1 + al_3d_2al_1 + al_3d_3al_1 - a_2al_2al_3al_1v_1 - al_3a_3v_1)x_3\\
&+ 2(al_3al_1 - 1)y_0 - 2(al_3 + al_1)y_1 - 2(al_1v_1 + al_3v_1)y_2 + 2(al_3al_1v_1 - v_1)y_3 = 0
\end{aligned}
$$

Step 3: if necessary add a base transformation -> general 3R-chain

All general 3R chains can be written without specifying the Denavit Hartenberg parameters

# Inverse kinematics of the general 6R-mechanism

$$\mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \mathbf{M}_3 \cdot \mathbf{G}_3 \cdot \mathbf{M}_4 \cdot \mathbf{G}_4 \cdot \mathbf{M}_5 \cdot \mathbf{G}_5 \cdot \mathbf{M}_6 \cdot \mathbf{G}_6 = \mathbf{A}$$

**A** is the given endeffector pose w.r.t. the base coordinate system

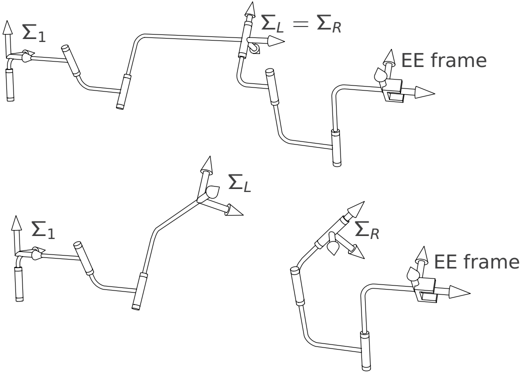


**Figure:** Cutting of the 6R into two 3R serial chains

Constraint variety of the left 3R-chain (= canonical 3-R chain):

$$\mathbf{T}_1 = \mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \mathbf{M}_3 \cdot \mathbf{G}_3.$$

Constraint variety of the right 3R-chain (= general 3R-chain):

$$\mathbf{T}_2 = \mathbf{A} \cdot \mathbf{G}_6^{-1} \cdot \mathbf{M}_6^{-1} \cdot \mathbf{G}_5^{-1} \cdot \mathbf{M}_5^{-1} \cdot \mathbf{G}_4^{-1} \cdot \mathbf{M}_4^{-1}.$$

**Theorem**

*Geometrically the solution of the inverse kinematic problem of a serial 6R-chain is equivalent to the intersection of eight one parameter sets of hyperplanes with $S_6^2$ in $P^7$.*

# Constraint Equations

- *Using geometric properties of the mechanism*

  properties can be for example: one point of the moving system (end effector system) is bound to move on a line, a circle, a sphere or a plane.

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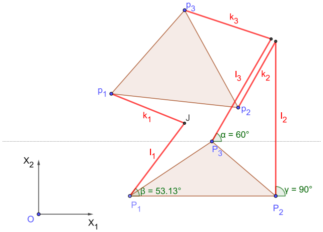
- *Linear implicitization algorithm (LIA)*

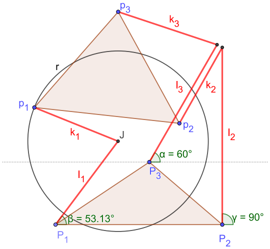
  guarantees a complete solution of the elimination.

  algorithm essentially solves an overconstrained linear system which can be very large in case of high degree polynomial constraint equations.

# Geometric constraint equations

Example: planar 3-RRR manipulator




$$X_1^2 + X_2^2 - 2mX_0X_1 - 2nX_0X_2 + (m^2 + n^2 - r^2)X_0^2 = 0$$

# Geometric constraint equations

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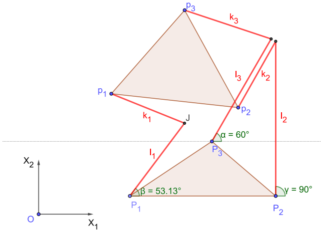
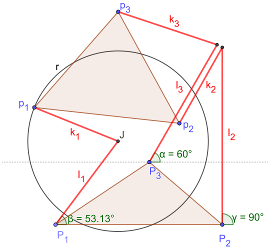
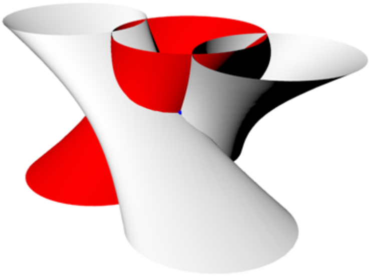



$$X_1^2 + X_2^2 - 2mX_0X_1 - 2nX_0X_2 + (m^2 + n^2 - r^2)X_0^2 = 0$$

$$(x^2 + y^2 + m^2 - 2mx + n^2 - 2ny - r^2)x_0^2 + 4(my - nx)x_0x_3 + 4(m - x)x_0y_1 +$$
$$4(n - y)x_0y_2 + (x^2 + y^2 + m^2 + 2mx + n^2 + 2ny - r^2)x_3^2 + 4(y + n)x_3y_1 -$$
$$4(x + m)x_3y_2 + 4y_1^2 + 4y_2^2 = 0.$$



$$P_1 = [1,0,0]^T, \quad P_2 = [1,A_2,0]^T, \quad P_3 = [1,A_3,B_3]^T,$$
$$p_1 = [1,0,0]^T, \quad p_2 = [1,a_2,0]^T, \quad p_3 = [1,a_3,b_3]^T.$$

Revolute input joints:

$$m_1 = l_1 \frac{1-u^2}{1+u^2}, \quad m_2 = l_2 \frac{1-v^2}{1+v^2} + A_2, \quad m_3 = l_3 \frac{1-w^2}{1+w^2} + A_3,$$
$$n_1 = l_1 \frac{2u}{1+u^2}, \quad n_2 = l_2 \frac{2v}{1+v^2}, \quad n_3 = l_3 \frac{2w}{1+w^2} + B_3.$$

$$h_1 : (l_1^2 - k_1^2)(x_0^2 + x_3^2) + 4l_1\left(\frac{1-u^2}{1+u^2}(x_0y_1 - x_3y_2) + \frac{2u}{1+u^2}(x_0y_2 + x_3y_1)\right) + 4(y_1^2 + y_2^2) = 0,$$

$$h_2 : \left(\frac{r_1r_2v^2 + r_3r_4)}{v^2+1}\right)x_0^2 + \left(\frac{r_5r_6v^2 + r_7r_8)}{v^2+1}\right)x_3^2 - 4a_2(x_0y_1 + x_3y_2) +$$

$$4(l_2\frac{1-v^2}{1+v^2} + A_2)(x_0y_1 - x_3y_2) + 4l_2\frac{2v}{1+v^2}(a_2x_0x_3 + x_0y_2 + x_3y_2) + 4(y_1^2 + y_2^2) = 0,$$

$$h_3 : \frac{(q_1^2 + q_2)w^2 + 4l_3(B_3 - b_3)w + q_4^2 + q_2q_3}{1+w^2}x_0^2 + \left(4\left(\frac{l_3(1-w^2)}{w^2+1} + A_3\right)b_3 - (4(\frac{2wl_3}{1+w^2} + B_3))a_3)\right)x_0x_3$$

$$\left(-4a_3 + 4l_3\frac{1-w^2}{w^2+1} + 4A_3\right)x_0y_1 + \left(-4b_3 + \frac{8wl_3}{(w^2+1} + 4B_3\right)x_0y_2 + \left(4b_3 + \frac{8wl_3}{w^2+1} + 4B_3\right)x_3y_1$$

$$+ \left(-4a_3 - 4l_3\frac{1-w^2}{w^2+1} - 4A_3\right)x_3y_2\frac{(q_5^2 + q_6q_7)w^2 + 4l_3(B_3b_3)w + q_8^2 + q_6q_7}{1+w^2}x_3^2 + 4(y_1^2 + y_2^2) = 0,$$

Using the three equations $h_1, h_2, h_3$ and a normalization condition one can solve the direct kinematics (DK), the inverse kinematics (IK), the forward and the inverse singularities completely.

The following design variables are assigned to a 3-RRR planar parallel manipulator:

$$A_2 = 16, A_3 = 9, B_3 = 6, a_2 = 14, a_3 = 7, b_3 = 10, l_1 = 10, l_2 = 17, l_3 = 13,$$
$$k_1 = \sqrt{75}, k_2 = \sqrt{70}, k_3 = 10.$$

Three input variables are given by

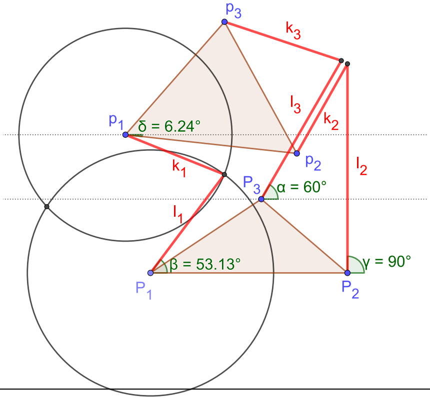
$$u = \frac{1}{2}, v = 1, w = \frac{\sqrt{3}}{3}.$$

Constraint equations simplify considerably

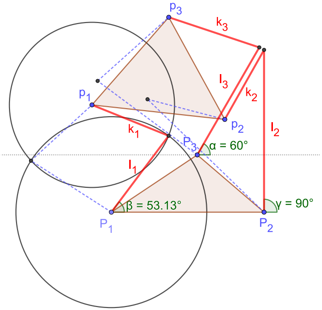
$$h_1 : 25x_3^2 + 32x_3y_1 - 24x_3y_2 + 4y_1^2 + 4y_2^2 + 24y_1 + 32y_2 + 25 = 0,$$
$$h_2 : 1119x_3^2 + 68x_3y_1 - 120x_3y_2 + 4y_1^2 + 4y_2^2 - 952x_3 + 8y_1 + 68y_2 + 223 = 0,$$
$$h_3 : 620\,x_3 + \frac{2025\,{x_3}^2}{4} - 130\sqrt{3} - \frac{191}{4} + 40\,y_1\,x_3 + 34\,y_1 - 90\,x_3\,y_2 - 40\,y_2 + 4\,{y_1}^2 +$$
$$4\,{y_2}^2 + \left(20{x_3}^2 + 4\,y_1x_3 - 28x_3 + 4y_2\right)\left(\frac{13\sqrt{3}}{2} + 6\right) + \left({x_3}^2 + 1\right)\left(\frac{13\sqrt{3}}{2} + 6\right)^2 = 0.$$

## Direct Kinematics:

$$1012018158645001\,x_3{}^6 + 373126531431576\,\sqrt{3}x_3{}^5 + 828170897821956\,\sqrt{3}x_3{}^4$$
$$-\,1870238901095276\,x_3{}^5 - 3830372502668712\,\sqrt{3}x_3{}^3 - 309592552617273\,x_3{}^4 -$$
$$1367698801300104\,\sqrt{3}x_3{}^2 + 5703740216839288\,x_3{}^3 + 2552443644341760\,\sqrt{3}x_3 +$$
$$2666944473586507\,x_3{}^2 - 584052482710476\,\sqrt{3} - 4438269370622172\,x_3 +$$
$$1009620776386125 = 0.$$



$$
\begin{aligned}
h_1 : {}& 25u^2x_3^2 + 40u^2x_3y_2 + 4u^2y_1^2 + 4u^2y_2^2 - 40u^2y_1 + 80ux_3y_1 + 25u^2 + 80uy_2 + 25x_3^2 - \\
& 40x_3y_2 + 4y_1^2 + 4y_2^2 + 40y_1 + 25 = 0 \\
h_2 : {}& 99v^2x_3^2 - 52v^2x_3y_2 + 4v^2y_1^2 + 4v^2y_2^2 - 60v^2y_1 + 136vx_3y_1 + 155v^2 - 1904vx_3 + \\
& 136vy_2 + 2139x_3^2 - 188x_3y_2 + 4y_1^2 + 4y_2^2 + 76y_1 + 291 = 0 \\
h_3 : {}& 165w^2x_3^2 + 64w^2x_3y_1 - 12w^2x_3y_2 + 4w^2y_1^2 + 4w^2y_2^2 - 328w^2x_3 - 44w^2y_1 - 16w^2y_2 + \\
& 832wx_3^2 + 104wx_3y_1 + 37w^2 - 728wx_3 + 104wy_2 + 997x_3^2 + 64x_3y_1 - 116x_3y_2 + 4y_1^2 + \\
& 4y_2^2 - 208w + 712x_3 + 60y_1 - 16y_2 + 14 = 0.
\end{aligned}
\tag{17}
$$



## Singularities

$$\mathbf{J}_o \dot{\mathbf{y}} + \mathbf{J}_i \dot{\mathbf{t}} = 0, \tag{18}$$

where

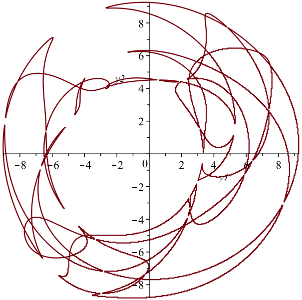
$$\mathbf{J}_o = \begin{bmatrix} \frac{\partial n}{\partial x_0} & \frac{\partial n}{\partial x_3} & 0 & 0 \\ \frac{\partial h_1}{\partial x_0} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial x_0} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \\ \frac{\partial h_3}{\partial x_0} & \frac{\partial h_3}{\partial x_3} & \frac{\partial h_3}{\partial y_1} & \frac{\partial h_3}{\partial y_2} \end{bmatrix}, \quad \mathbf{J}_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\partial h_1}{\partial u} & 0 & 0 \\ 0 & 0 & \frac{\partial h_2}{\partial v} & 0 \\ 0 & 0 & 0 & \frac{\partial h_3}{\partial w} \end{bmatrix},$$

Forward singularities: $\dot{\mathbf{t}} = [0, 0, 0, 0]^T$

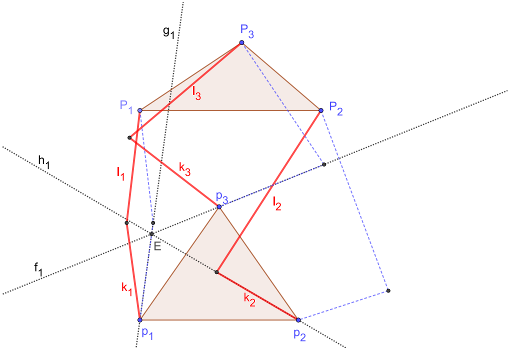
$$\mathbf{J}_o \dot{\mathbf{y}} = 0.$$

Determinant of $\mathbf{J}_o \rightarrow h_4 = 0$ polynomial of degree 10 in the unknowns $x_0, x_3, y_1, y_2, u, v, w \rightarrow$
$h_1, h_2, h_3, h_4$ system of four algebraic equations
elimination of $u, v, w$ yields a polynomial of degree 44 which describes all forward singularities

one could also eliminate the Study parameters and would get the forward singularities in joint space



y2
y1

g1
P3
l3
P1
P2
h1
k3
l1
p3
l2
E
f1
k1
k2
p1
p2

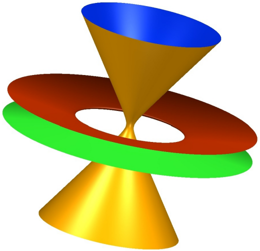
Inverse singularities:

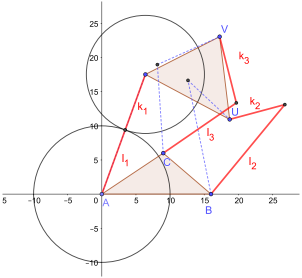
$$\mathbf{J}_i \dot{\mathbf{t}} = 0.$$

It is quite obvious that this determinant factors into three parts:

$$\begin{aligned}
h_5 : \big[&(B_3x_0^2 + B_3x_3^2 - 2a_3x_0x_3 - b_3x_0^2 + b_3x_3^2 + 2x_0y_2 + 2x_3y_1)w^2 + \\
&(2A_3x_0^2 + 2A_3x_3^2 - 2a_3x_0^2 + 2a_3x_3^2 + 4b_3x_0x_3 + 4x_0y_1 - 4x_3y_2)w \\
&-2x_0y_2 - 2x_3y_1 - B_3x_0^2 - B_3x_3^2 + 2a_3x_0x_3 + b_3x_0^2 - b_3x_3^2\big]\, l_3 \cdot \\
&\big[(-a_2x_0x_3 + x_0y_2 + x_3y_1)v^2 + (A_2x_0^2 + A_2x_3^2 - a_2x_0^2 + a_2x_3^2)v + \\
&a_2x_0x_3 + 2vx_0y_1 - 2vx_3y_2 - x_0y_2 - x_3y_1)l_2\big] \cdot \big[(u^2x_0y_2 + u^2x_3y_1 + 2ux_0y_1 - 2ux_3y_2 - x_0y_2 - x_3y_1)l_1\big] = 0.
\end{aligned}$$

In kinematic image space:





Inverse singularities in joint space:

system of equations: $\mathcal{S} = \{h_1, h_2, h_3, h_5\}$ in $x_0, x_3, y_1, y_2, u, v, w$ eliminate Study parameters!

result is equation of degree 28 in $u, v, w$.

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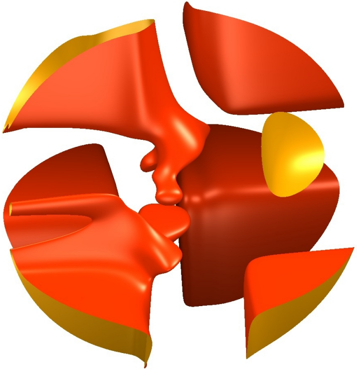
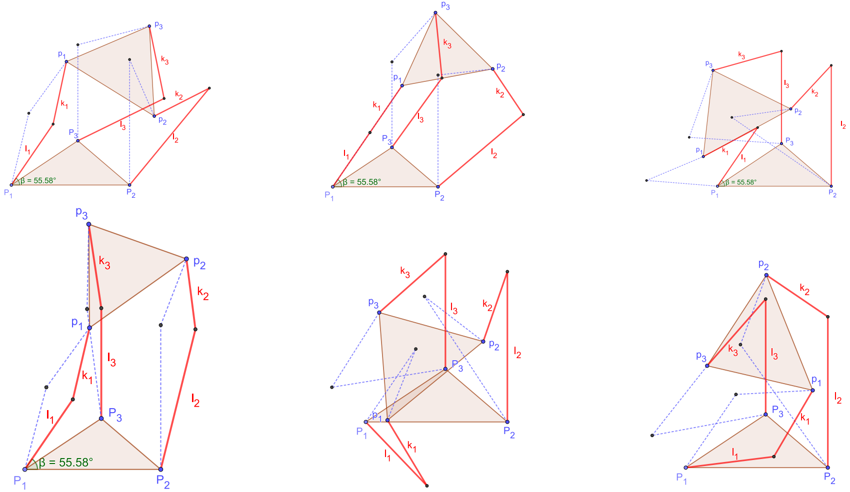


# Compute one point on singularity surface and from this the pose of the manipulator!



## *Elimination Method*

simple recipe: Write the forward kinematics of the kinematic chain and than eliminate the motion parameters

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number $m$ of constraint equations (in general ) to be expected is $m = 6 - n$.

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Example:

$$\mathbf{l} = [1 : u : v : uv : -uav : av : ua : -a].$$

homogeneous vector equation consists of eight component equations

$$\rho x_0 = 1, \rho x_1 = u, \rho x_2 = v, \rho x_3 = uv, \rho y_0 = -auv, \rho y_1 = av, \rho y_2 = au, \rho y_3 = -a.$$

eliminate the motion parameters $u$ and $v$

$$x_3 - x_1 x_2 = 0, \quad y_0 + a x_1 x_2 = 0, \quad y_1 - a x_2 = 0, \quad y_2 - a x_1 = 0, \quad y_3 + a = 0.$$

five?
manipulation and observing that the Study quadric has to be fulfilled yields

$$y_0 + a x_3 = 0, \quad y_1 - a x_2 = 0, \quad y_2 - a x_1 = 0, \quad y_3 + a = 0.$$

# Linear Implicitization Algorithm (LIA)

Is there an algorithm that derives “automatically” from a parametric representation of the (allowed) kinematic chain a minimal set of implicit equations that completely describes this kinematic chain?

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Two basic ideas:

- a kinematic chain built from only revolute and prismatic joints can be represented by a set of polynomials
- the parametric expressions have to fulfill the polynomial equations

- there exists a one-to-one correspondence from all spatial transformations to the Study quadric

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- determine $C_k$
- possibly increase the degree of the ansatz polynomial

Example: Canonical leg of a Stewart-Gough platform (UPS-chain)

Denavit-Hartenberg parameters:

| | $\alpha_i$ | $a_i$ | $d_i$ |
|---|---|---|---|
| $\mathbf{G}_1$ | $\frac{\pi}{2}$ | 0 | 0 |
| $\mathbf{G}_2$ | 0 | L | 0 |
| $\mathbf{G}_3$ | $\frac{\pi}{2}$ | 0 | 0 |
| $\mathbf{G}_4$ | $\frac{\pi}{2}$ | 0 | 0 |

- Write the forward kinematics of the canonical chain

$$\mathbf{D} = \mathbf{M}_1 \cdot \mathbf{G}_1 \cdot \mathbf{M}_2 \cdot \mathbf{G}_2 \cdot \mathbf{M}_3 \cdot \mathbf{G}_3 \cdot \mathbf{M}_4 \cdot \mathbf{G}_4 \cdot \mathbf{M}_5.$$

- perform half-tangent substitution to make the equations algebraic.

$$
\begin{aligned}
x_0 = & -1 + t_5t_1 - t_5t_2 - t_5t_1t_2t_3 + t_2t_5t_1t_4 - t_1t_4 - t_5t_4 - t_5t_3 + t_4t_1t_2t_3 + t_4t_2t_3t_5 - t_1t_2 - \\
& t_4t_3 - t_4t_2 + t_4t_1t_3t_5 - t_1t_3 + t_2t_3 \\
x_1 = & -t_4t_1t_2t_3 - t_5t_1t_2t_3 - t_2t_5t_1t_4 - t_1t_2 - t_4t_1t_3t_5 - t_1t_3 + t_1t_4 + t_5t_1 + t_4t_2t_3t_5 - t_2t_3 \\
& - t_4t_2 + t_5t_2 - t_4t_3 + 1 + t_5t_3 - t_5t_4 \\
x_2 = & t_1 + t_2 - t_1t_2t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 - t_4 + t_5t_1t_2 + t_3 + t_2t_5t_4 + t_4t_2t_3 + t_5t_2t_3 - t_4t_1t_3 \\
& - t_5 + t_5t_1t_3 - t_5t_1t_4 + t_4t_3t_5 \\
x_3 = & -t_1 + t_2 + t_1t_2t_3 - t_5t_1t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 + t_4 - t_5t_1t_2 + t_3 - t_5t_1t_4 - t_4t_2t_3 - \\
& t_4t_1t_3 - t_5 + t_5t_2t_3 - t_2t_5t_4 - t_4t_3t_5 \\
y_0 = & \ldots
\end{aligned}
$$

$$
\begin{aligned}
x_0 = & -1 + t_5t_1 - t_5t_2 - t_5t_1t_2t_3 + t_2t_5t_1t_4 - t_1t_4 - t_5t_4 - t_5t_3 + t_4t_1t_2t_3 + t_4t_2t_3t_5 - t_1t_2 - \\
& t_4t_3 - t_4t_2 + t_4t_1t_3t_5 - t_1t_3 + t_2t_3 \\
x_1 = & -t_4t_1t_2t_3 - t_5t_1t_2t_3 - t_2t_5t_1t_4 - t_1t_2 - t_4t_1t_3t_5 - t_1t_3 + t_1t_4 + t_5t_1 + t_4t_2t_3t_5 - t_2t_3 \\
& - t_4t_2 + t_5t_2 - t_4t_3 + 1 + t_5t_3 - t_5t_4 \\
x_2 = & \, t_1 + t_2 - t_1t_2t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 - t_4 + t_5t_1t_2 + t_3 + t_2t_5t_4 + t_4t_2t_3 + t_5t_2t_3 - t_4t_1t_3 \\
& - t_5 + t_5t_1t_3 - t_5t_1t_4 + t_4t_3t_5 \\
x_3 = & -t_1 + t_2 + t_1t_2t_3 - t_5t_1t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 + t_4 - t_5t_1t_2 + t_3 - t_5t_1t_4 - t_4t_2t_3 - \\
& t_4t_1t_3 - t_5 + t_5t_2t_3 - t_2t_5t_4 - t_4t_3t_5 \\
y_0 = & \, \ldots
\end{aligned}
$$

- Make a general Ansatz polynomial in Study coordinates.
- Substitute the above equations.
- Order with respect to the $t_i$.

$$
\begin{aligned}
x_0 =\ & -1 + t_5t_1 - t_5t_2 - t_5t_1t_2t_3 + t_2t_5t_1t_4 - t_1t_4 - t_5t_4 - t_5t_3 + t_4t_1t_2t_3 + t_4t_2t_3t_5 - t_1t_2 - \\
& t_4t_3 - t_4t_2 + t_4t_1t_3t_5 - t_1t_3 + t_2t_3 \\
x_1 =\ & -t_4t_1t_2t_3 - t_5t_1t_2t_3 - t_2t_5t_1t_4 - t_1t_2 - t_4t_1t_3t_5 - t_1t_3 + t_1t_4 + t_5t_1 + t_4t_2t_3t_5 - t_2t_3 \\
& - t_4t_2 + t_5t_2 - t_4t_3 + 1 + t_5t_3 - t_5t_4 \\
x_2 =\ & t_1 + t_2 - t_1t_2t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 - t_4 + t_5t_1t_2 + t_3 + t_2t_5t_4 + t_4t_2t_3 + t_5t_2t_3 - t_4t_1t_3 \\
& - t_5 + t_5t_1t_3 - t_5t_1t_4 + t_4t_3t_5 \\
x_3 =\ & -t_1 + t_2 + t_1t_2t_3 - t_5t_1t_3 - t_4t_1t_2 + t_1t_4t_2t_3t_5 + t_4 - t_5t_1t_2 + t_3 - t_5t_1t_4 - t_4t_2t_3 - \\
& t_4t_1t_3 - t_5 + t_5t_2t_3 - t_2t_5t_4 - t_4t_3t_5 \\
y_0 =\ & \dots
\end{aligned}
$$

- Make a general Ansatz polynomial in Study coordinates.
- Substitute the above equations.
- Order with respect to the $t_i$.

$$
\begin{aligned}
& (C_3L + C_1L + 2C_4 - 2C_2)t_1 + (-C_7L + 2C_6 + C_5L + 2C_8)t_4t_2t_3t_5 \\
& + (C_7L + C_5L + 2C_8 - 2C_6)t_4t_1t_3t_5 + \ldots + (C_3L + C_1L + 2C_4 - 2C_2)t_4t_2t_3 = 0.
\end{aligned}
$$

$$
\begin{aligned}
x_0 = & -1 + t_5 t_1 - t_5 t_2 - t_5 t_1 t_2 t_3 + t_2 t_5 t_1 t_4 - t_1 t_4 - t_5 t_4 - t_5 t_3 + t_4 t_1 t_2 t_3 + t_4 t_2 t_3 t_5 - t_1 t_2 - \\
& t_4 t_3 - t_4 t_2 + t_4 t_1 t_3 t_5 - t_1 t_3 + t_2 t_3 \\
x_1 = & -t_4 t_1 t_2 t_3 - t_5 t_1 t_2 t_3 - t_2 t_5 t_1 t_4 - t_1 t_2 - t_4 t_1 t_3 t_5 - t_1 t_3 + t_1 t_4 + t_5 t_1 + t_4 t_2 t_3 t_5 - t_2 t_3 \\
& - t_4 t_2 + t_5 t_2 - t_4 t_3 + 1 + t_5 t_3 - t_5 t_4 \\
x_2 = & \, t_1 + t_2 - t_1 t_2 t_3 - t_4 t_1 t_2 + t_1 t_4 t_2 t_3 t_5 - t_4 + t_5 t_1 t_2 + t_3 + t_2 t_5 t_4 + t_4 t_2 t_3 + t_5 t_2 t_3 - t_4 t_1 t_3 \\
& - t_5 + t_5 t_1 t_3 - t_5 t_1 t_4 + t_4 t_3 t_5 \\
x_3 = & -t_1 + t_2 + t_1 t_2 t_3 - t_5 t_1 t_3 - t_4 t_1 t_2 + t_1 t_4 t_2 t_3 t_5 + t_4 - t_5 t_1 t_2 + t_3 - t_5 t_1 t_4 - t_4 t_2 t_3 - \\
& t_4 t_1 t_3 - t_5 + t_5 t_2 t_3 - t_2 t_5 t_4 - t_4 t_3 t_5 \\
y_0 = & \ldots
\end{aligned}
$$

- Make a general Ansatz polynomial in Study coordinates.
- Substitute the above equations.
- Order with respect to the $t_i$.

$$
\begin{aligned}
&(C_3 L + C_1 L + 2C_4 - 2C_2) t_1 + (-C_7 L + 2C_6 + C_5 L + 2C_8) t_4 t_2 t_3 t_5 \\
&+ (C_7 L + C_5 L + 2C_8 - 2C_6) t_4 t_1 t_3 t_5 + \ldots + (C_3 L + C_1 L + 2C_4 - 2C_2) t_4 t_2 t_3 = 0.
\end{aligned}
$$

Solve the linear system

$$
(y_0^2 + y_1^2 + y_2^2 + y_3^2 - \frac{1}{4} L^2 (x_0^2 + x_1^2 + x_2^2 + x_3^2))\lambda + (x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3)\mu = 0.
$$

how do we obtain the constraint equations of a chain in general position from the constraint equations of a kinematic chain in canonical position?

M. Pfurner. Analysis of spatial serial manipulators using kinematic mapping. PhD thesis, University of Innsbruck, 2006. URL http://repository.uibk.ac.at.

answer:

Changes of coordinate systems in base and end-effector coordinate system induce linear transformations of the Study coordinates

important consequence:

These transformations make the equations more complicated but do not change their degree!!!

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_1 & 1 & 0 & 0 \\ B_1 & 0 & 1 & 0 \\ C_1 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 \\ -b_1 & 0 & 1 & 0 \\ -c_1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_1 & 1 & 0 & 0 \\ B_1 & 0 & 1 & 0 \\ C_1 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_1 & 1 & 0 & 0 \\ -b_1 & 0 & 1 & 0 \\ -c_1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{T}_m\mathbf{T}_f = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & -2\,a1 + 2\,A_1 & -2\,b_1 + 2\,B_1 & -2\,c_1 + 2\,C_1 & 4 & 0 & 0 & 0 \\ 2\,a_1 - 2\,A_1 & 0 & 2\,c_1 + 2\,C_1 & -2\,b_1 - 2\,B_1 & 0 & 4 & 0 & 0 \\ 2\,b_1 - 2\,B_1 & -2\,c_1 - 2\,C_1 & 0 & 2\,a_1 + 2\,A_1 & 0 & 0 & 4 & 0 \\ 2\,c_1 - 2\,C_1 & 2\,b_1 + 2\,B_1 & -2\,a_1 - 2\,A_1 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Applying these transformations to the canonical chain yields

$$
\begin{aligned}
&\left((a_1 - A_1)\,x_1' + (b_1 - B_1)\,x_2' + (c_1 - C_1)\,x_3' + 2\,y_0'\right)^2 + \\
&\left((-a_1 + A_1)\,x_0' + (-c_1 - C_1)\,x_2' + (b_1 + B_1)\,x_3' + 2\,y_1'\right)^2 + \\
&\left((-b_1 + B_1)\,x_0' + (c_1 + C_1)\,x_1' + (-a_1 - A_1)\,x_3' + 2\,y_2'\right)^2 + \\
&\left((-c_1 + C_1)\,x_0' + (-b_1 - B_1)\,x_1' + (a_1 + A_1)\,x_2' + 2\,y_3'\right)^2 - \\
&\frac{1}{4}\,L^2\left(4\,{x_0'}^2 + 4\,{x_1'}^2 + 4\,{x_2'}^2 + 4\,{x_3'}^2\right) = 0.
\end{aligned}
$$