GDR robotique

Robot Modelling

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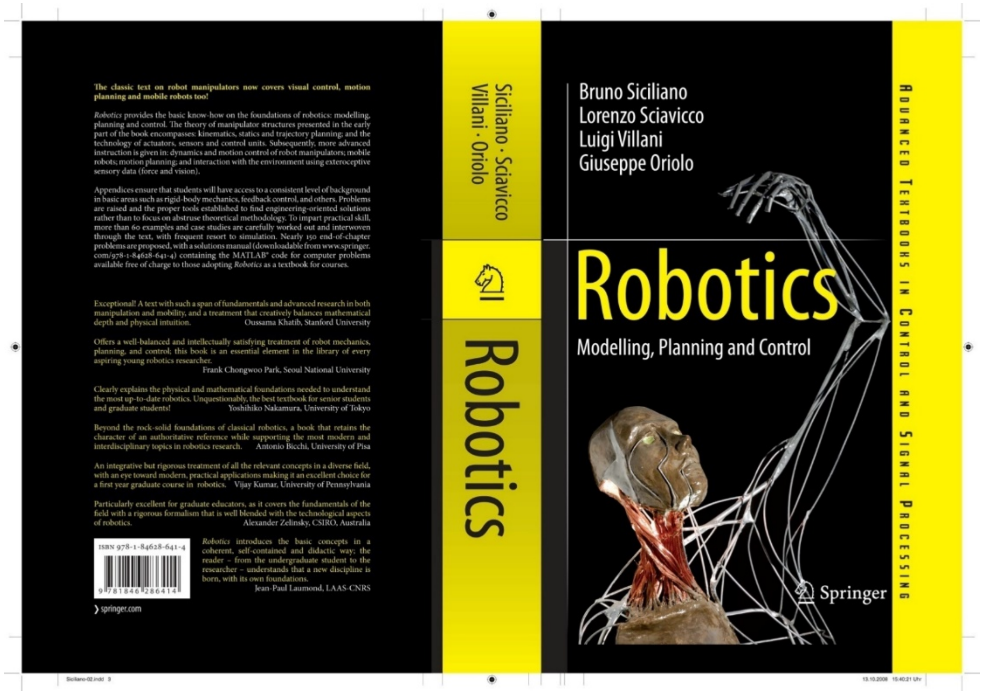
www.prisma.unina.it

- Robots and robotics
- Kinematics
- Differential Kinematics
- Statics
- Dynamics

The Textbook

B. Siciliano, L. Sciavicco, L. Villani, G. Oriolo, *Robotics: Modelling, Planning and Control*, Springer, London, 2009, DOI 10.1007/978-1-4471-0449-0

- Chapter 1 — Introduction
- Chapter 2 — Kinematics
- Chapter 3 — Differential Kinematics and Statics
- Chapter 7 — Dynamics



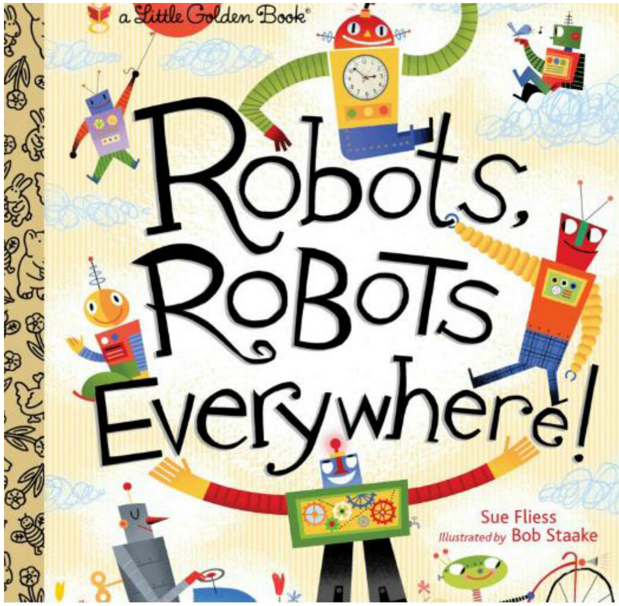
MOOC Robotics Foundations – Robot Modelling

https://www.federica.eu/c/robotics_foundations_i_robot_modelling

B. Siciliano, O. Khatib, *Springer Handbook of Robotics 2nd Edition*, Springer, Heidelberg, 2016, DOI 10.1007/978-3-319-32552-1

- Chapter 2 — Kinematics
- Chapter 3 — Dynamics
- Chapter 4 — Mechanisms and Actuation

Robots !!!



Mars

Oceans

Hospitals

Factories

Schools

Homes

...

Today

Tomorrow

Intelligent

Personal

Pervasive

Disappearing

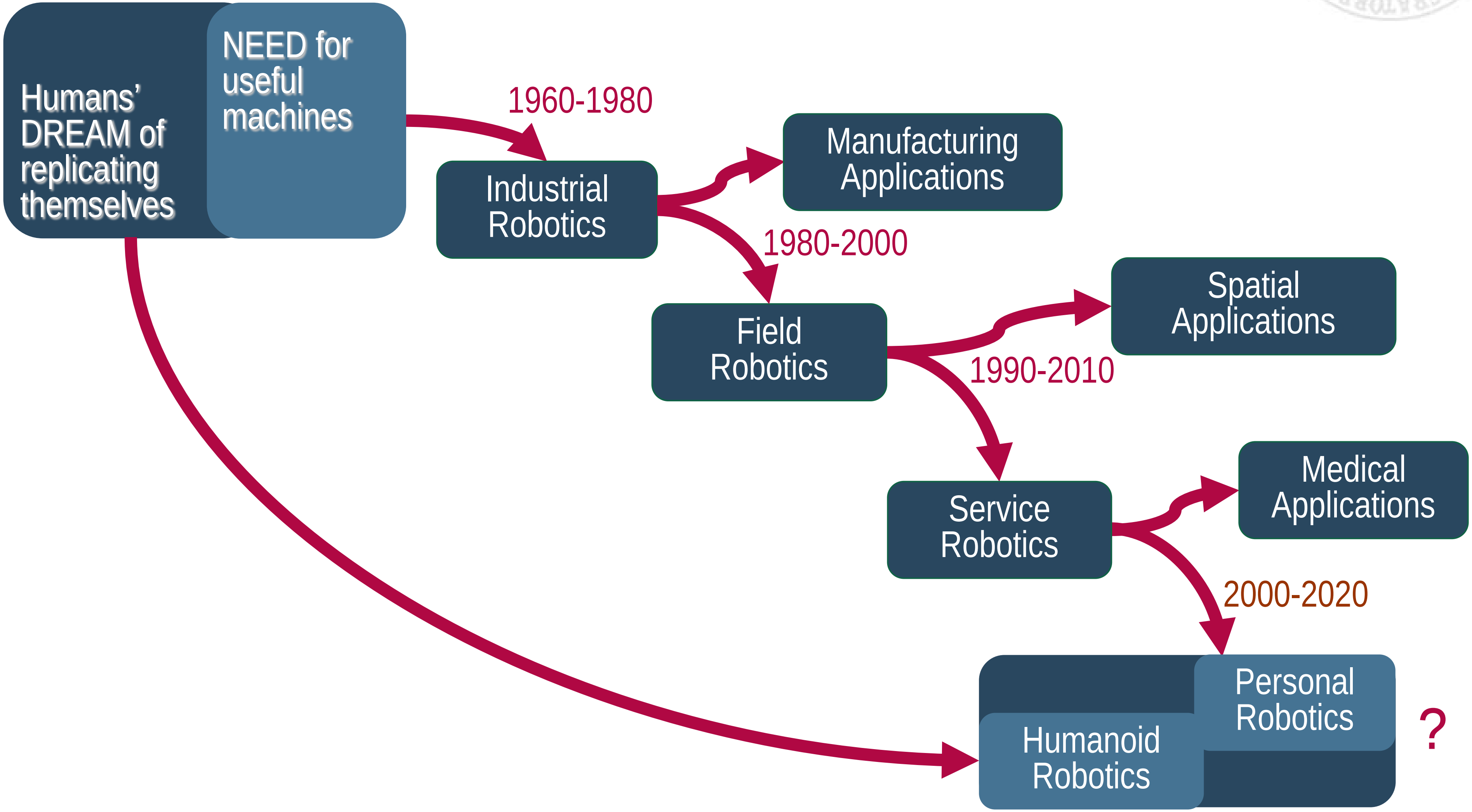
Ubiquitous

What is a Robot?

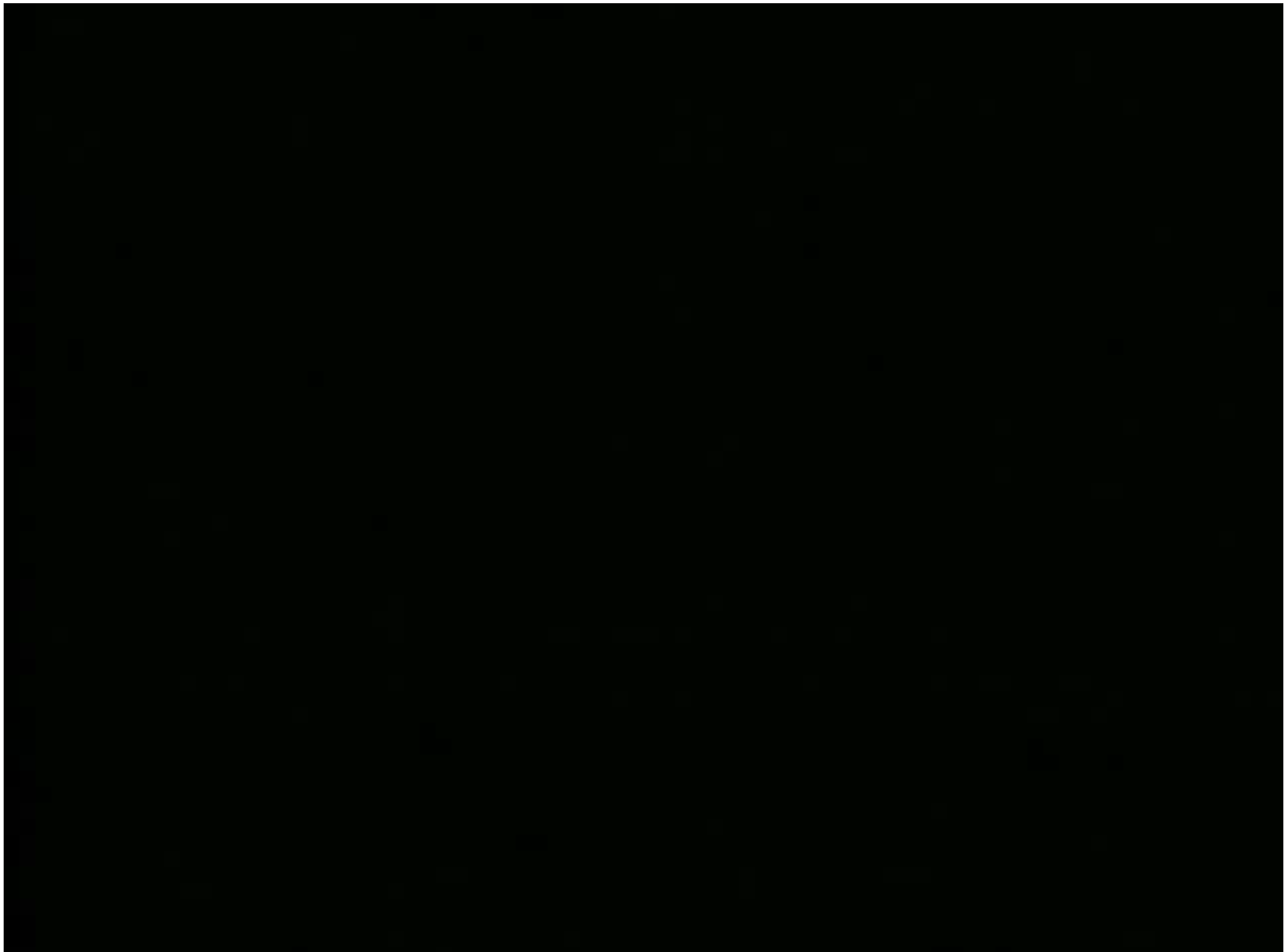
- Robot (robota = subordinate labour)
- One of humans' greatest ambitions has been to give life to their artifacts (mythology)
- Common people continue to imagine the robot as an android who can speak, walk, see, and hear, with an appearance very much like that of humans (science fiction)
- The robot is seen as a machine that, independently of its exterior, is able to execute tasks in an automatic way to replace or improve human labour (reality)

What is a Robot?

Evolution of Robotics

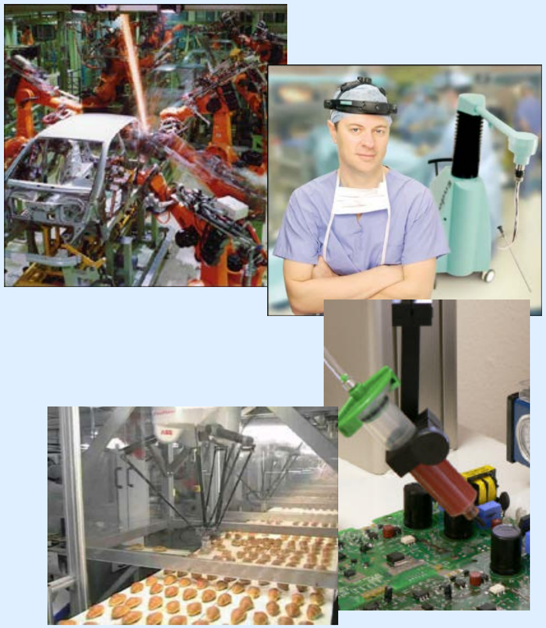


A 50 Year Journey



From Factories to Our Homes

Industry



Automotive
Chemical
Electronics
Food

Field

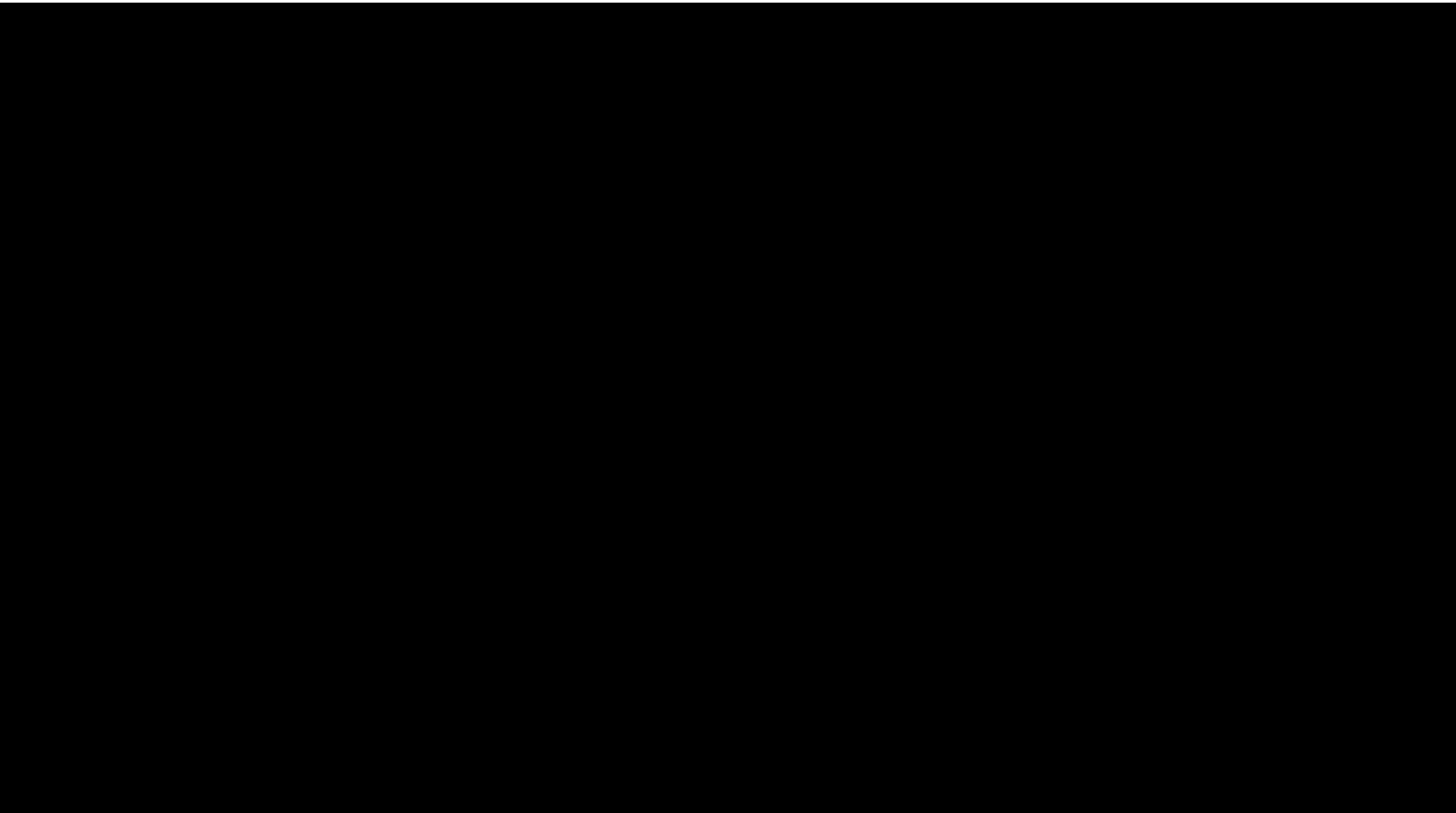
Aerial
Space
Underwater
Search and rescue

Service

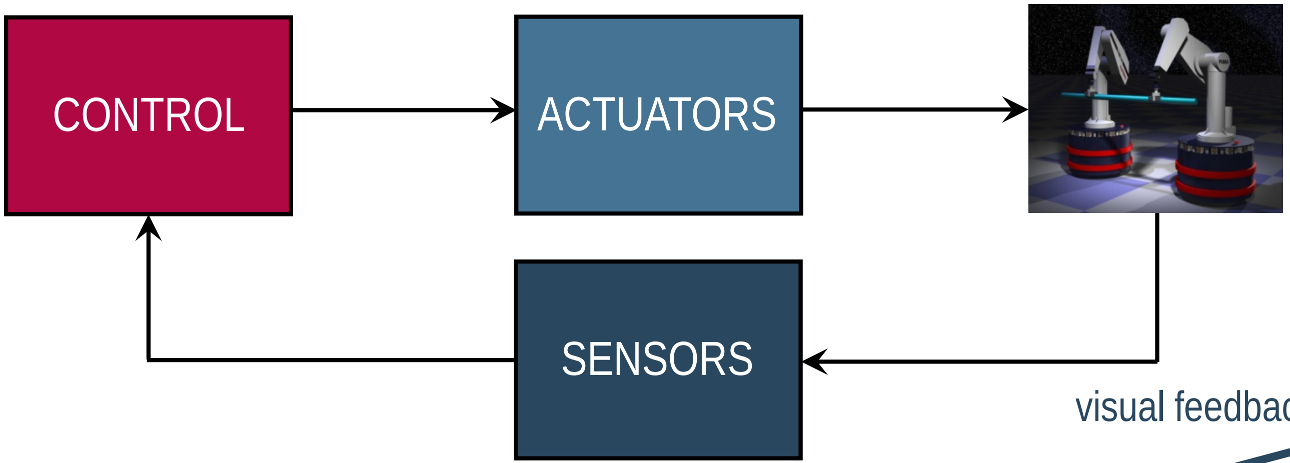
Domestic
Edutainment
Rehabilitation
Medical

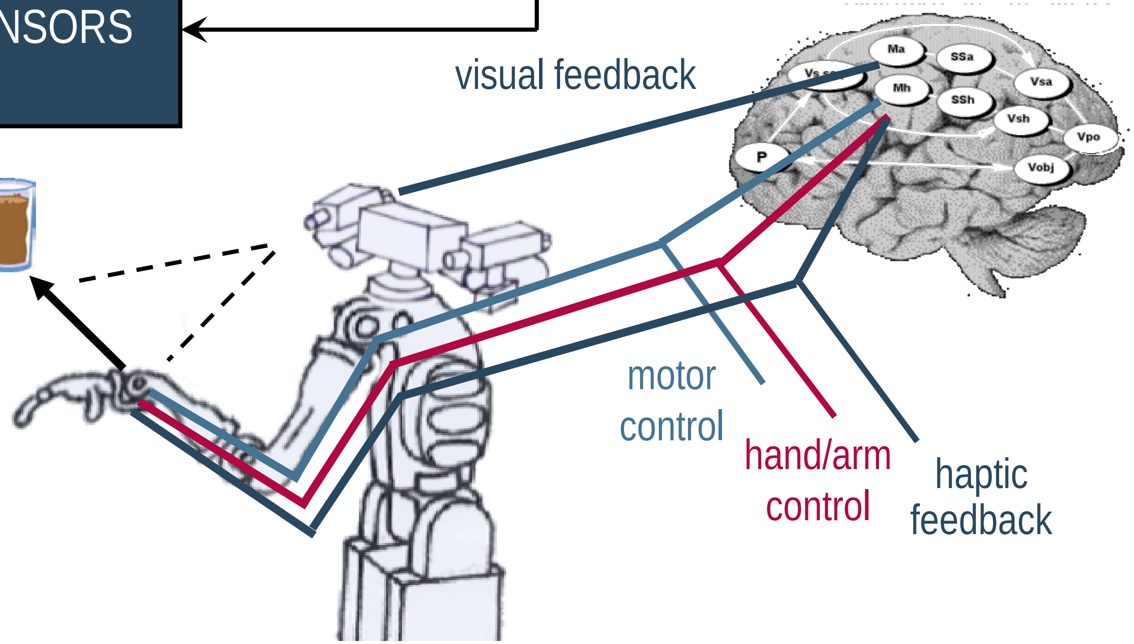
Level of Autonomy

The Journey Continues



intelligent connection between perception and action



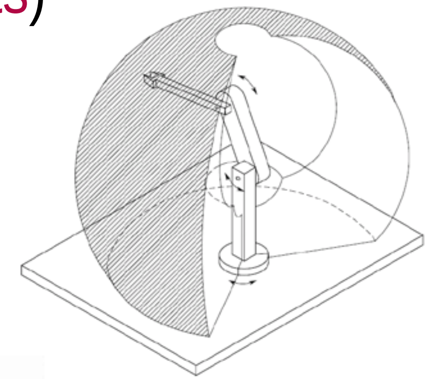


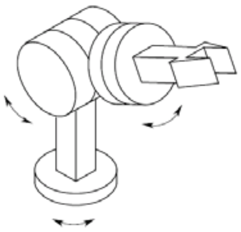
Components of a Robotic System

- Mechanical system
  - Locomotion apparatus (wheels, crawlers, mechanical legs)
  - Manipulation apparatus (mechanical arms, end-effectors, artificial hands)
- Actuation system
  - Animates the mechanical components of the robot
  - Motion control (servomotors, drives, transmissions
- Sensory system
  - Proprioceptive sensors (internal information on system)
  - Exteroceptive sensors (external information on environment)
- Control system
  - Execution of action set by task planning coping with robot and environment's constraints
  - Adoption of feedback principle
  - Use of system models

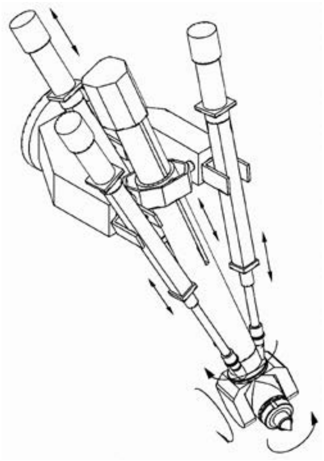
Robot Mechanical Structure

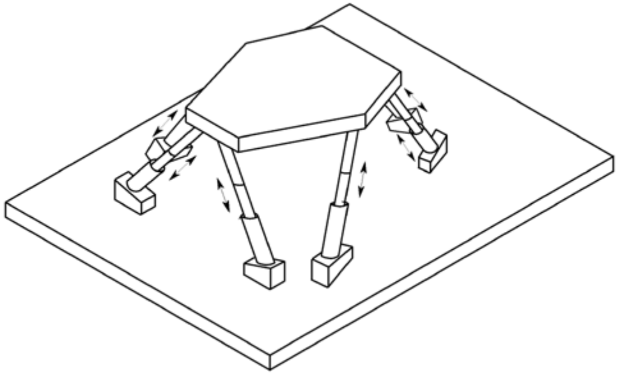
- Mechanical structure of robot manipulator: sequence of rigid bodies (links) interconnected by means of articulations (joints)
  - Arm ensuring mobility
  - Wrist conferring dexterity
  - End-effector performing the task required of robot





- Mechanical structure
  - Open vs. closed kinematic chain





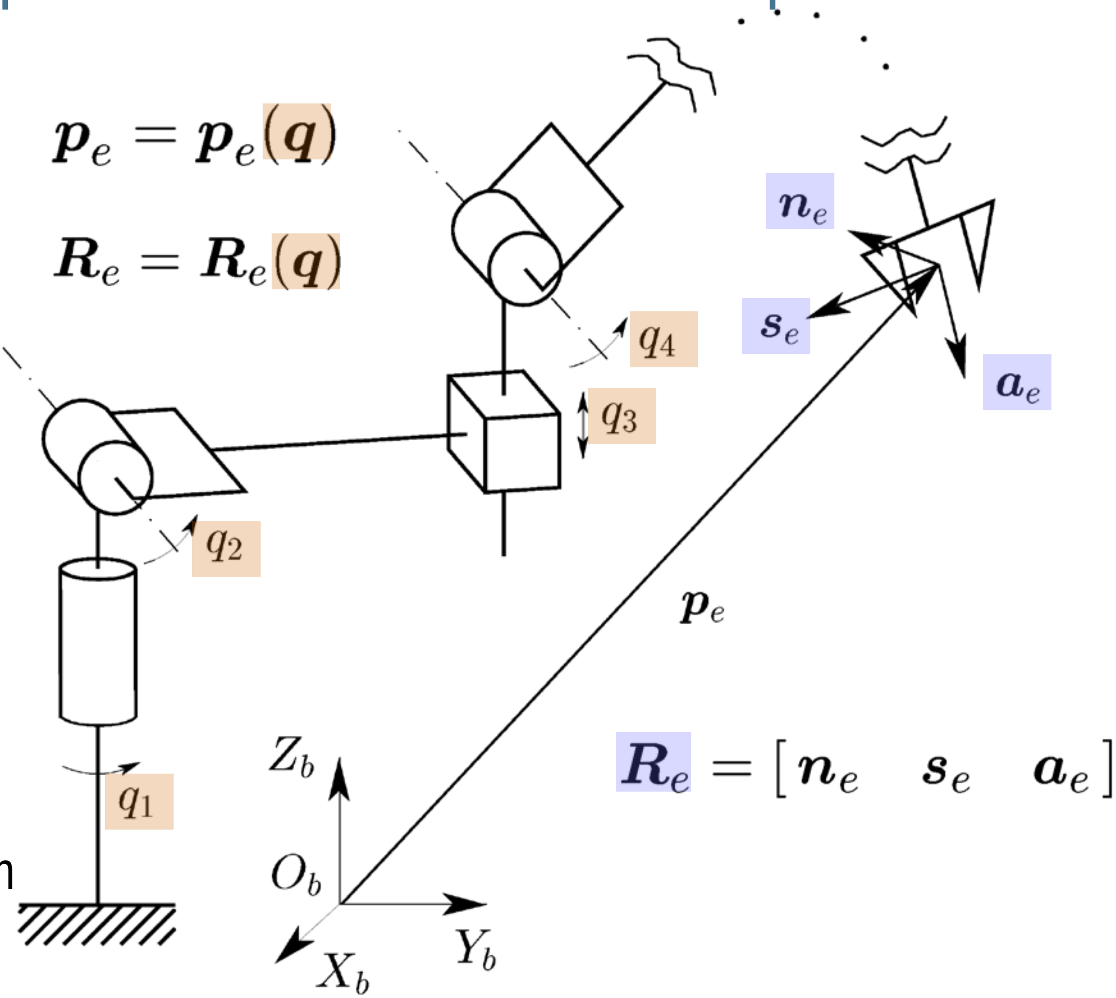
- Mobility
  - Prismatic vs. revolute joints
- Degrees of freedom
  - 3 for position + 3 for orientation
- Workspace
  - Portion of environment the manipulator's end-effector can access

Relationship between the joint positions and the end-effector pose

- Representations of orientation
  - Rotation matrix
  - Euler angles
  - Four-parameter representations
- Direct kinematics
  - Homogeneous transformations
  - Denavit-Hartenberg convention
  - Examples
- Inverse kinematics
  - Solution of three-link planar arm
  - Solution of anthropomorphic arm
  - Solution of spherical wrist

$$\boldsymbol{p}_e = \boldsymbol{p}_e(\boldsymbol{q})$$

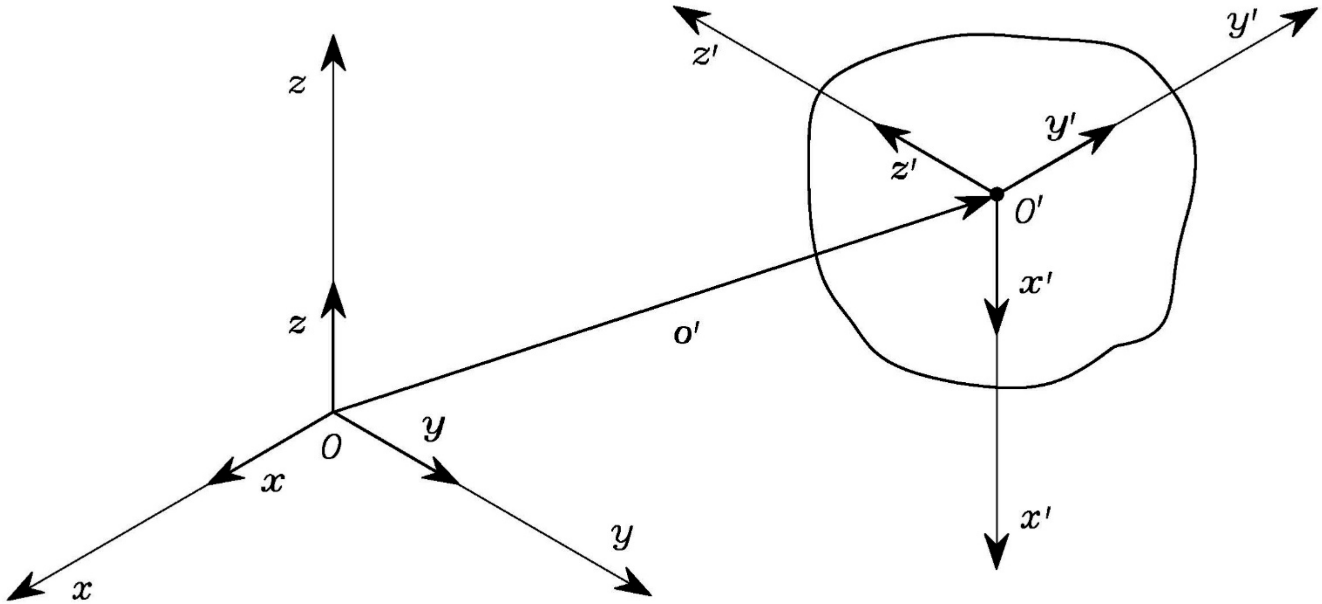
$$\boldsymbol{R}_e = \boldsymbol{R}_e(\boldsymbol{q})$$



$$\boldsymbol{R}_e = [\boldsymbol{n}_e \quad \boldsymbol{s}_e \quad \boldsymbol{a}_e]$$

- Position

$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$



- Orientation

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$$

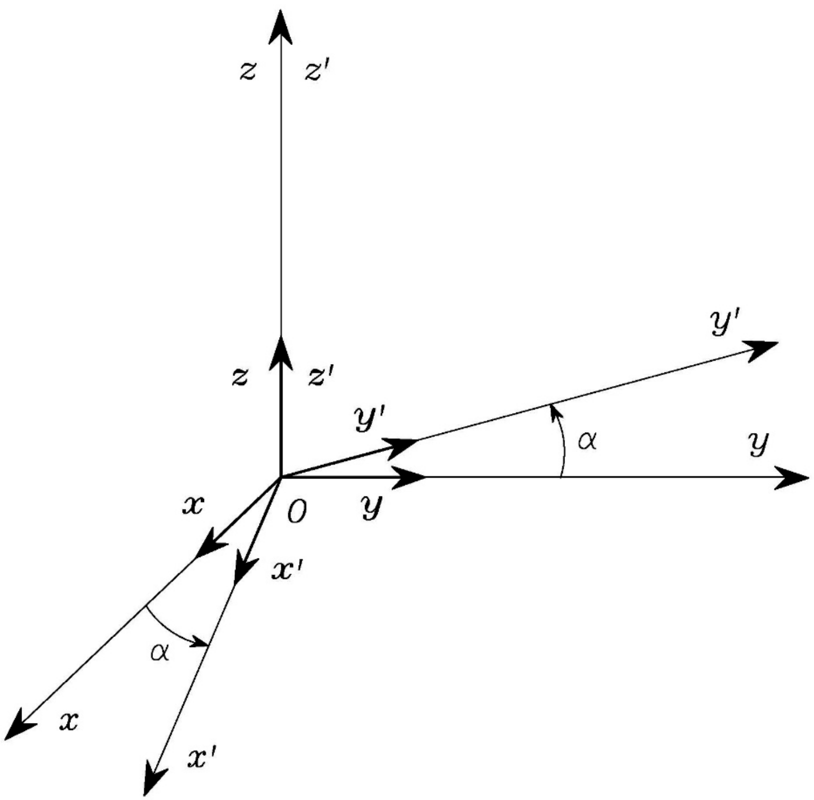
$$\boldsymbol{R}^T = \boldsymbol{R}^{-1}$$

Elementary Rotations

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$



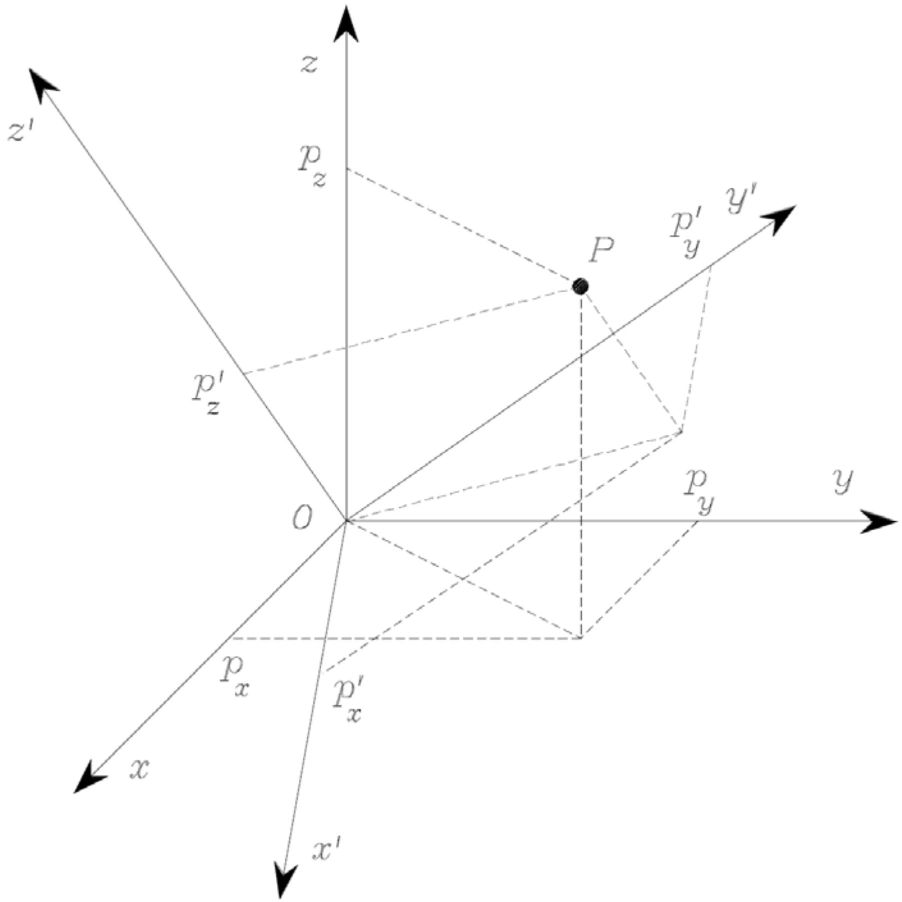
Representation of a Vector

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

$$= \boldsymbol{R}\boldsymbol{p}'$$

$$\boldsymbol{p}' = \boldsymbol{R}^T \boldsymbol{p}$$

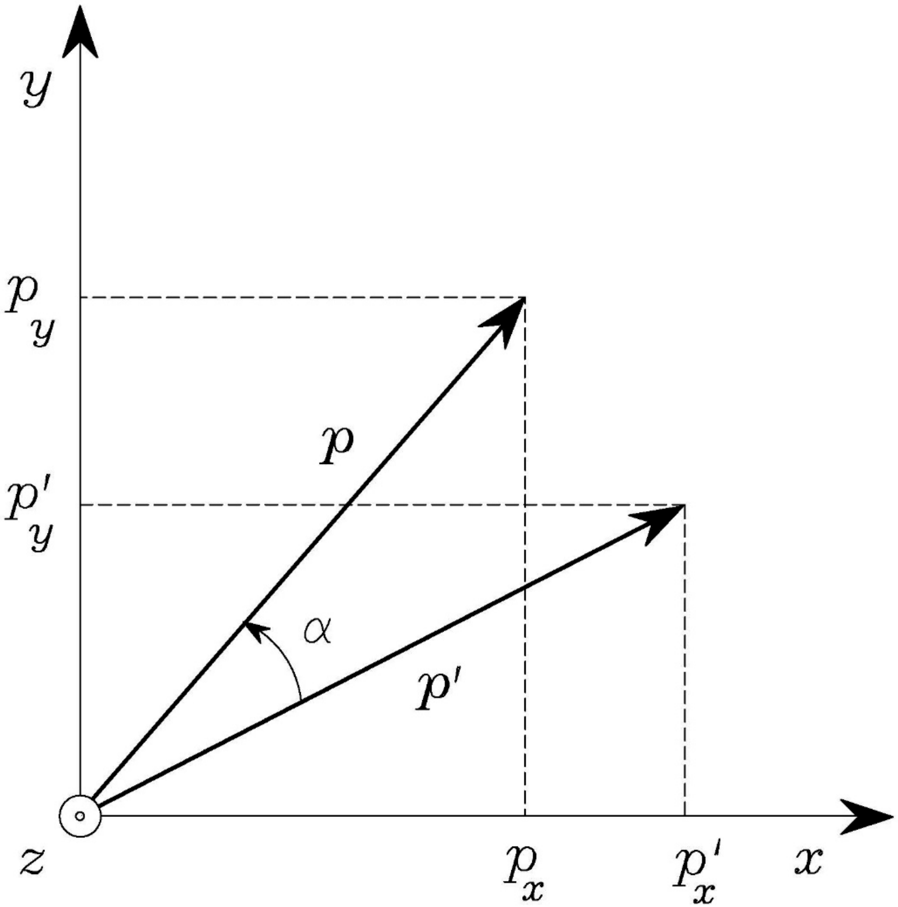


Rotation of a Vector

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'$$

$$\boldsymbol{p}^T\boldsymbol{p} = \boldsymbol{p}'^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{p}'$$

$$\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'$$



Three equivalent geometrical meanings

- It describes the mutual orientation between two coordinate frames; its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame
- It represents the coordinate transformation between the coordinates of a point expressed in two different frames (with common origin)
- It is the operator that allows the rotation of a vector in the same coordinate frame

Composition of Rotation Matrices

- Rotations in current frame

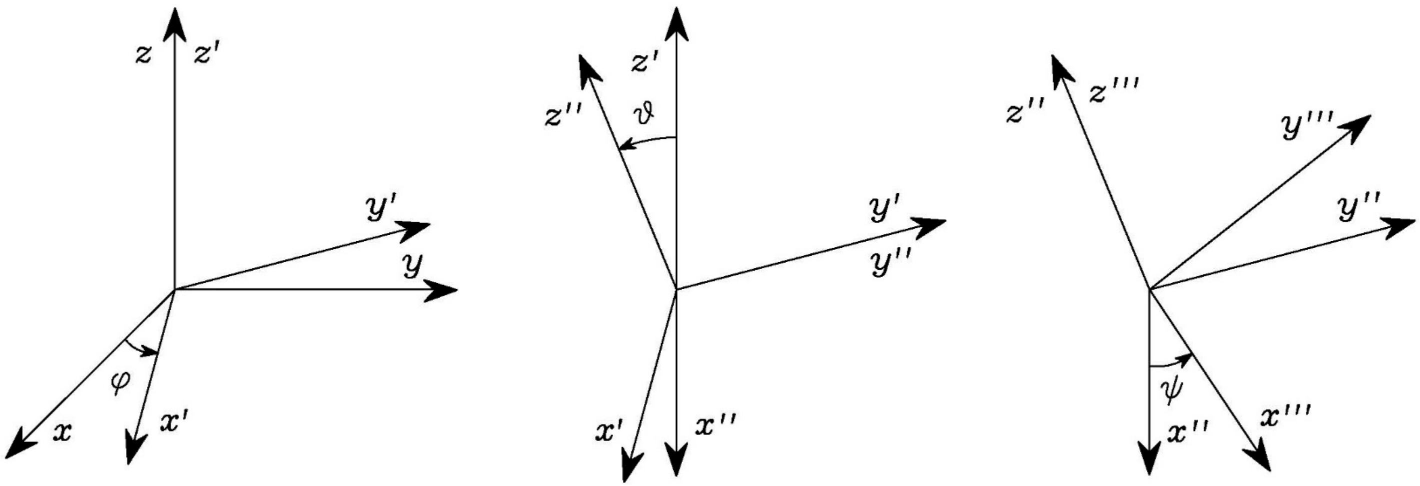
$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2$$

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$

- Rotation matrix
  - 9 parameters with 6 constraints
- Minimal representation of orientation
  - 3 independent parameters



$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

Inverse Problem

- Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Solution

$$\varphi = \text{Atan2}(r_{23}, r_{13})$$
$$\vartheta = \text{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$
$$\psi = \text{Atan2}(r_{32}, -r_{31})$$

$$\vartheta \in (0, \pi)$$

$$\varphi = \text{Atan2}(-r_{23}, -r_{13})$$
$$\vartheta = \text{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$
$$\psi = \text{Atan2}(-r_{32}, r_{31})$$

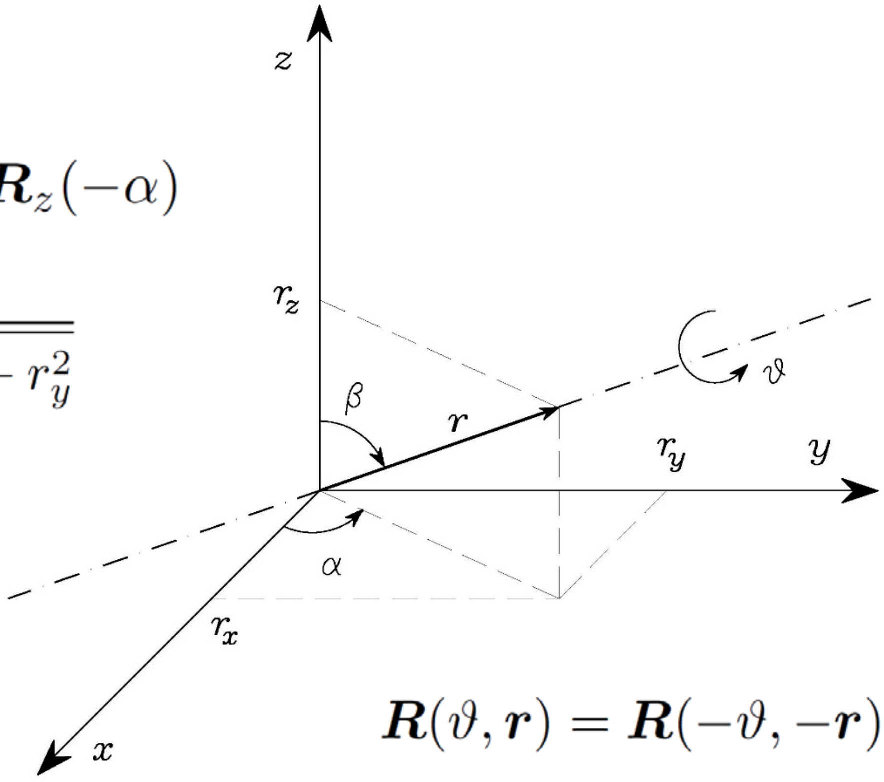
$$\vartheta \in (-\pi, 0)$$

- Four-parameter representation

$$\boldsymbol{R}(\vartheta,\boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha)$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2+r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2+r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2+r_y^2} \qquad \cos\beta = r_z$$



$$\boldsymbol{R}(\vartheta,\boldsymbol{r}) = \boldsymbol{R}(-\vartheta,-\boldsymbol{r})$$

$$\boldsymbol{R}(\vartheta,\boldsymbol{r}) = \begin{bmatrix} r_x^2(1-c_\vartheta)+c_\vartheta & r_xr_y(1-c_\vartheta)-r_zs_\vartheta & r_xr_z(1-c_\vartheta)+r_ys_\vartheta \\ r_xr_y(1-c_\vartheta)+r_zs_\vartheta & r_y^2(1-c_\vartheta)+c_\vartheta & r_yr_z(1-c_\vartheta)-r_xs_\vartheta \\ r_xr_z(1-c_\vartheta)-r_ys_\vartheta & r_yr_z(1-c_\vartheta)+r_xs_\vartheta & r_z^2(1-c_\vartheta)+c_\vartheta \end{bmatrix}$$

- Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Solution

$$\vartheta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$$\sin\vartheta \neq 0$$

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

- Four-parameter representation

$$\mathcal{Q} = \{\eta, \boldsymbol{\epsilon}\} \qquad \eta = \cos\frac{\vartheta}{2} \qquad \boldsymbol{\epsilon} = \sin\frac{\vartheta}{2}\boldsymbol{r} \qquad \eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$$

$$\boldsymbol{R}(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

- $(\vartheta, \boldsymbol{r})$ and $(-\vartheta, -\boldsymbol{r})$ give the same quaternion
- Quaternion extracted from $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$: $\mathcal{Q}^{-1} = \{\eta, -\boldsymbol{\epsilon}\}$
- Quaternion product: $\mathcal{Q}_1 * \mathcal{Q}_2 = \{\eta_1\eta_2 - \boldsymbol{\epsilon}_1^T\boldsymbol{\epsilon}_2, \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2\}$

- Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Solution

$$\eta = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1} \qquad \eta \geq 0$$

$$\boldsymbol{\epsilon} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

Homogeneous Representation of a Vector

- Coordinate transformation (translation + rotation)

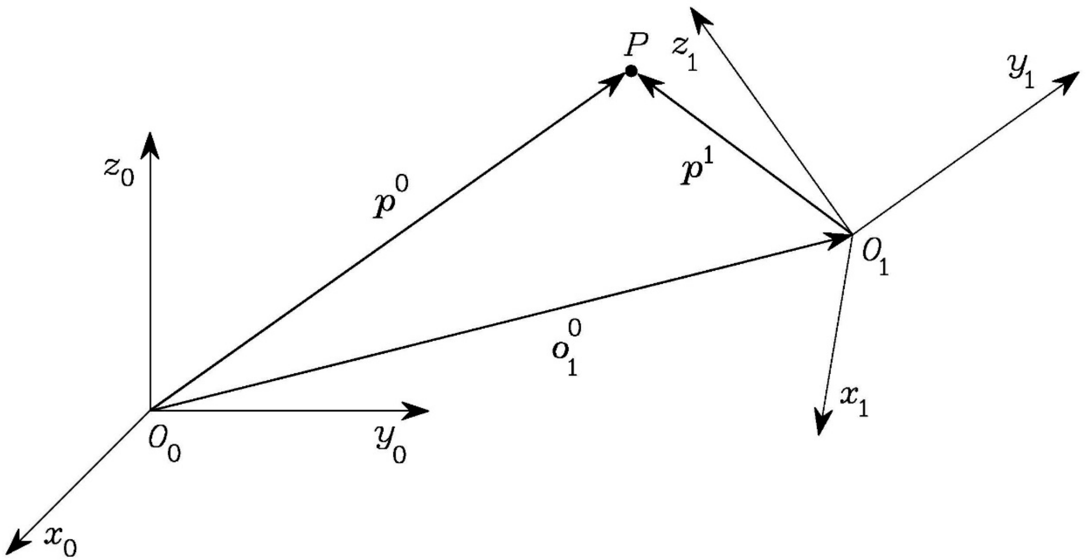
$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- Inverse transformation

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$

- Homogenous representation

$$\tilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$



Homogeneous Transformation Matrix

- Coordinate transformation

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1 \qquad \boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

- Inverse transformation

$$\tilde{\boldsymbol{p}}^1 = \boldsymbol{A}_0^1 \tilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \tilde{\boldsymbol{p}}^0 \qquad \boldsymbol{A}_0^1 = \begin{bmatrix} \boldsymbol{R}_0^1 & -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

- Orthogonality does not hold

$$\boldsymbol{A}^{-1} \neq \boldsymbol{A}^T$$

- Sequence of coordinate transformations

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \dots \boldsymbol{A}_n^{n-1} \tilde{\boldsymbol{p}}^n$$

Manipulator

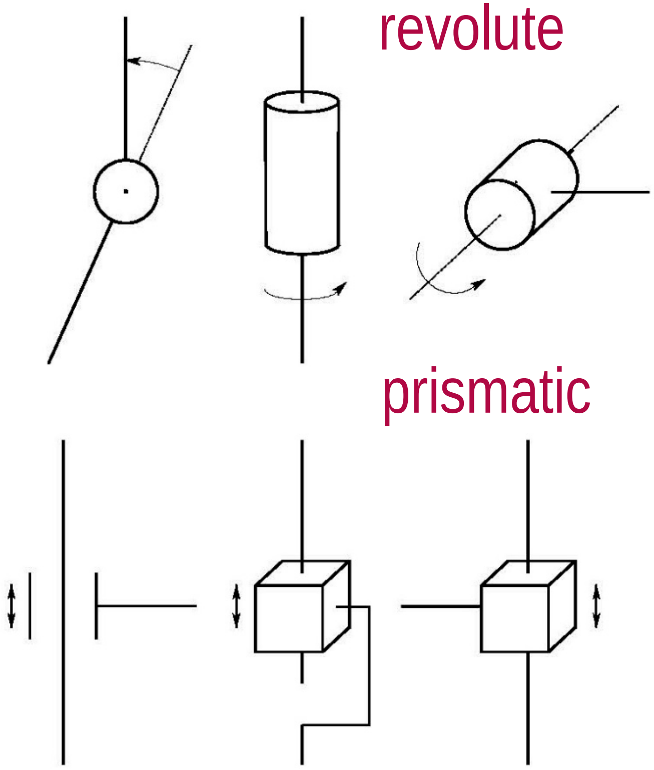
- Series of rigid bodies (links) connected by means of kinematic pairs or joints

Kinematic chain (from base to end-effector)

- Open (only one sequence of links connecting the two ends of the chain)
- Closed (a sequence of links forms a loop)

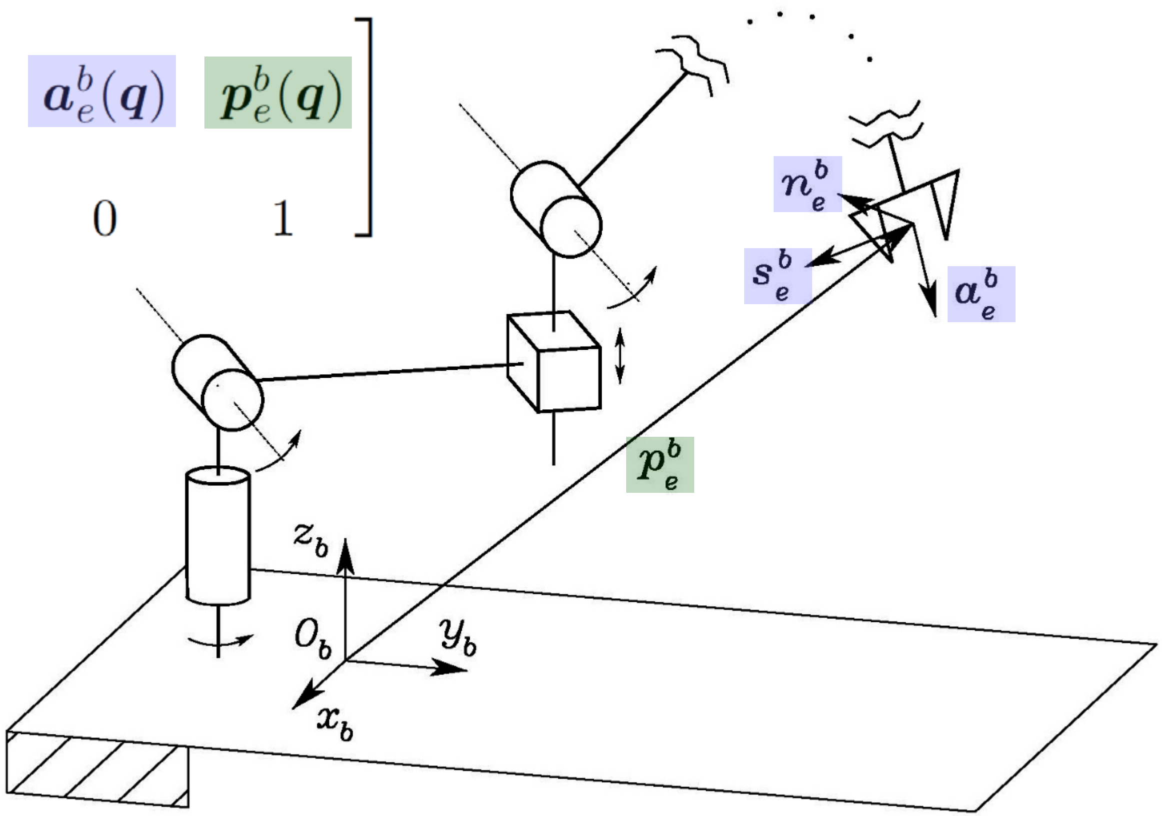
Degrees of freedom (DOFs) uniquely determine the manipulator's posture

- Each DOF is typically associated with a joint articulation and constitutes a joint variable



- End-effector frame with respect to base frame

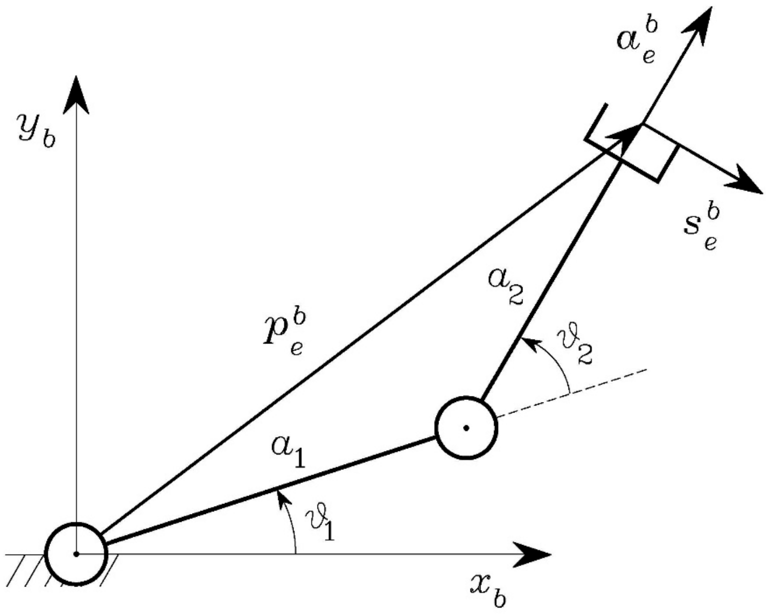
$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{n}_e^b(\boldsymbol{q}) & \boldsymbol{s}_e^b(\boldsymbol{q}) & \boldsymbol{a}_e^b(\boldsymbol{q}) & \boldsymbol{p}_e^b(\boldsymbol{q}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Two-Link Planar Arm

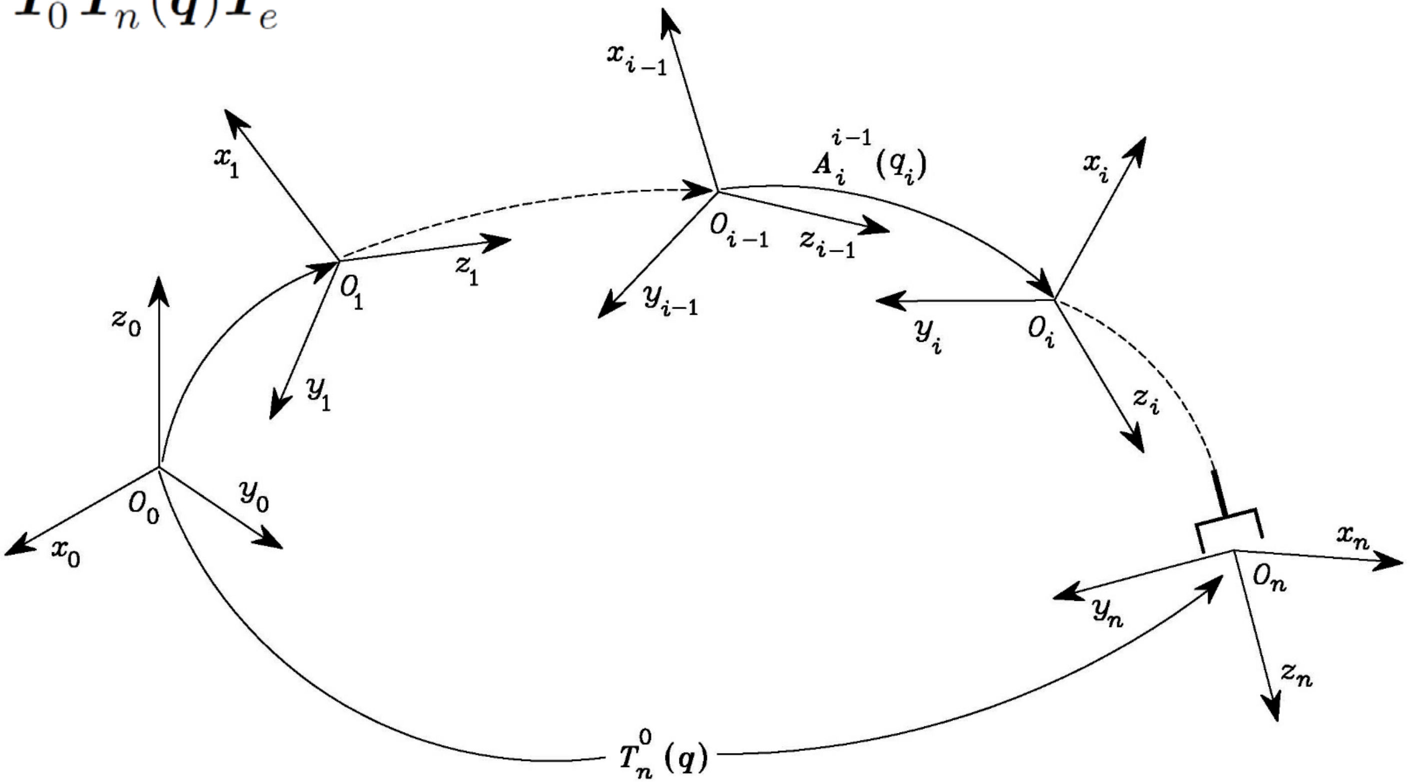
$$T_e^b(q) = \begin{bmatrix} n_e^b & s_e^b & a_e^b & p_e^b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_{12} & c_{12} & a_1c_1 + a_2c_{12} \\ 0 & -c_{12} & s_{12} & a_1s_1 + a_2s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



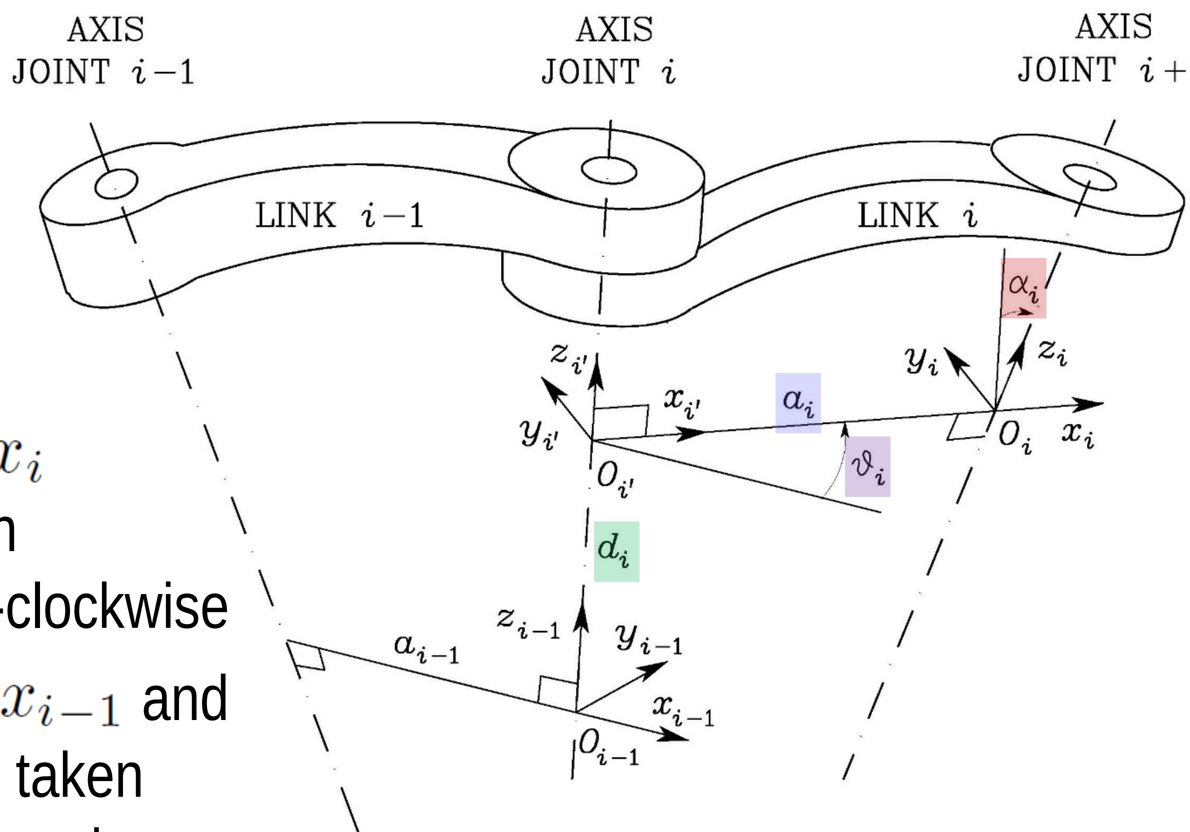
$$T_n^0(\mathbf{q}) = A_1^0(q_1)A_2^1(q_2)\dots A_n^{n-1}(q_n)$$

$$T_e^b(\mathbf{q}) = T_0^b T_n^0(\mathbf{q}) T_e^n$$



Denavit-Hartenberg Parameters

- a_i distance between O_i and O'_i
- d_i coordinate of O'_i along z_{i-1}
- α_i angle between axes z_{i-1} and z_i about axis x_i to be taken positive when rotation is made counter-clockwise
- ϑ_i angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive when rotation is made counter-clockwise



Coordinate Transformation

$$\boldsymbol{A}_{i'}^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{i}^{i-1}(q_i) = \boldsymbol{A}_{i'}^{i-1}\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c_{\alpha_i} & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

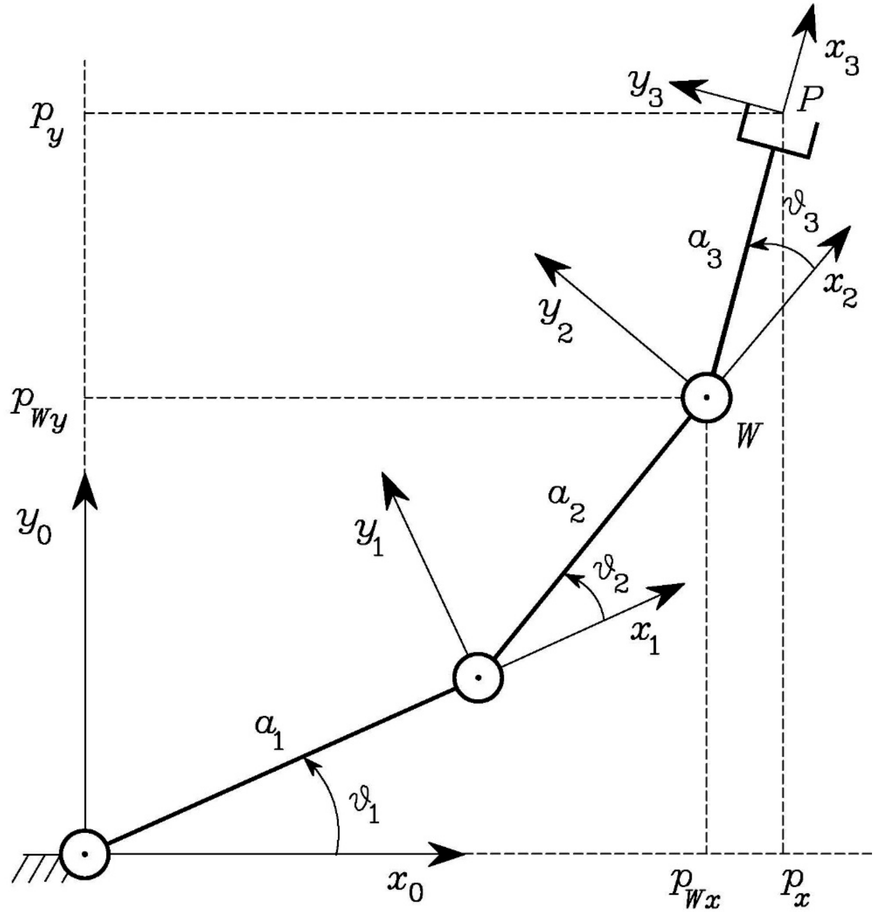
Three-Link Planar Arm

Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

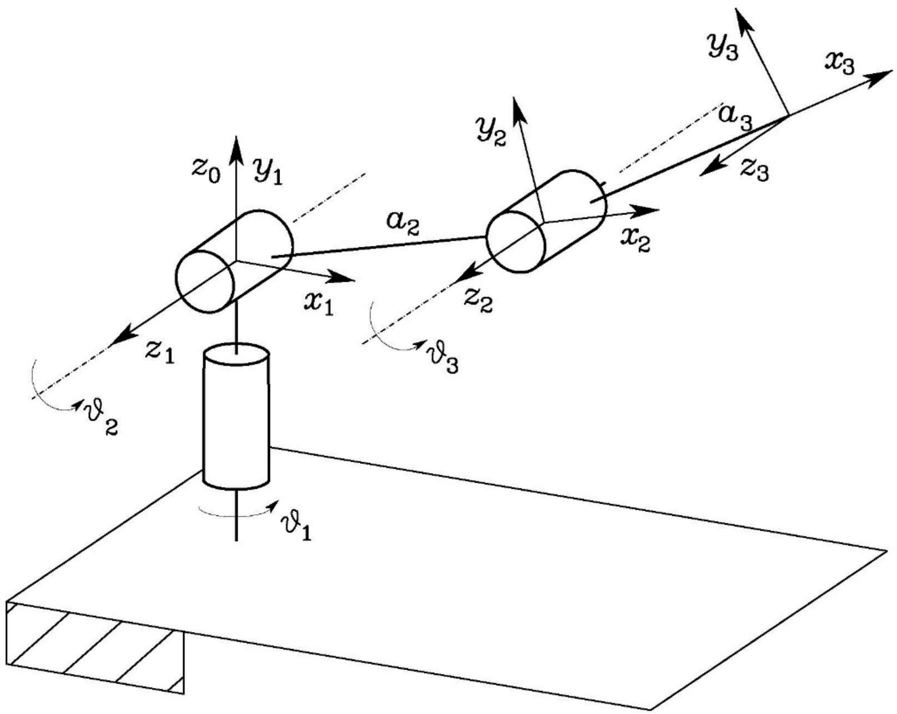
$$\boldsymbol{A}_i^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3$$

$$\boldsymbol{T}_3^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2$$

$$= \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

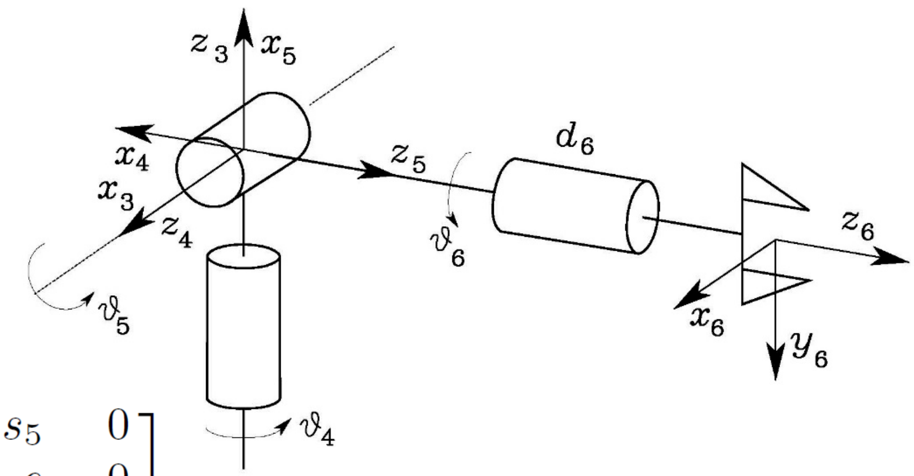


$$\boldsymbol{A}_1^0 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_i^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3$$

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6



$$A_4^3 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_5^4 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^5 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^3 = A_4^3 A_5^4 A_6^5$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

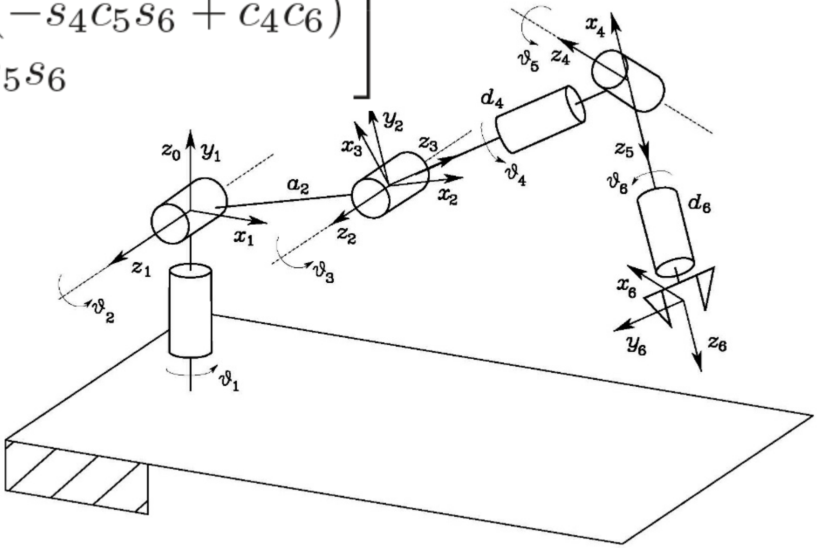
Anthropomorphic Arm with Spherical Wrist

$$\boldsymbol{p}_6^0 = \begin{bmatrix} a_2c_1c_2 + d_4c_1s_{23} + d_6\big(c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5\big) \\ a_2s_1c_2 + d_4s_1s_{23} + d_6\big(s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5\big) \\ a_2s_2 - d_4c_{23} + d_6(s_{23}c_4s_5 - c_{23}c_5) \end{bmatrix}$$

$$\boldsymbol{n}_6^0 = \begin{bmatrix} c_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) + s_1(s_4c_5c_6 + c_4s_6) \\ s_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) - c_1(s_4c_5c_6 + c_4s_6) \\ s_{23}(c_4c_5c_6 - s_4s_6) + c_{23}s_5c_6 \end{bmatrix}$$

$$\boldsymbol{s}_6^0 = \begin{bmatrix} c_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) + s_1(-s_4c_5s_6 + c_4c_6) \\ s_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) - c_1(-s_4c_5s_6 + c_4c_6) \\ -s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \end{bmatrix}$$

$$\boldsymbol{a}_6^0 = \begin{bmatrix} c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5 \\ s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5 \\ s_{23}c_4s_5 - c_{23}c_5 \end{bmatrix}$$



Joint space

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

- $q_i = \vartheta_i$ (revolute joint)
- $q_i = d_i$ (prismatic joint)

Operational space

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix} \qquad \begin{matrix} (m \times 1) \\ m \leq n \end{matrix}$$

Direct Kinematics Equation

$$\boldsymbol{x}_e = \boldsymbol{k}(\boldsymbol{q})$$

- $m < n$: kinematically redundant manipulator $\qquad m \leq 6$

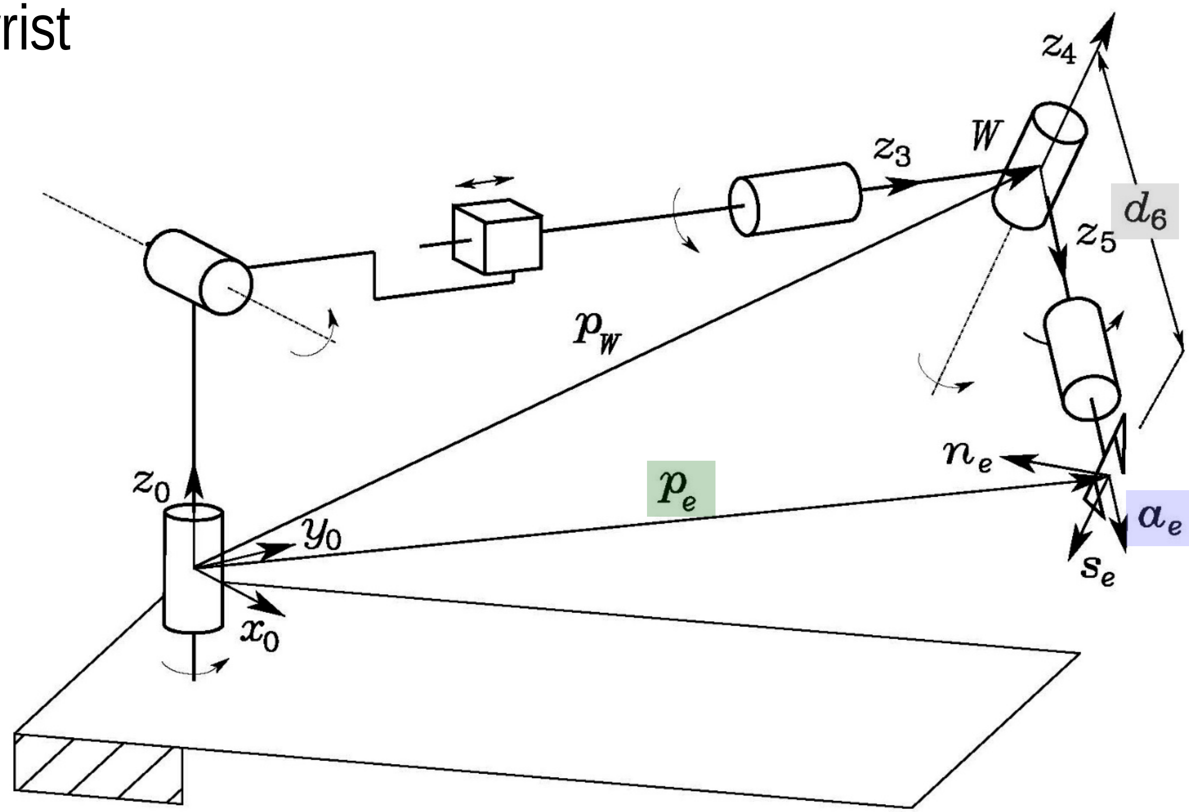
Inverse Kinematics

- Complexity
  - Possibility to find closed-form solutions (nonlinear equations to solve)
  - Existence of multiple solutions
  - Existence of infinite solutions (kinematically redundant manipulator)
  - No admissible solutions, in view of the manipulator kinematic structure
- Computation of closed-form solutions
  - Algebraic intuition
  - Geometric intuition
- No closed-form solutions
  - Numerical solution techniques

Kinematic Decoupling

Manipulators with spherical wrist

$$\boldsymbol{p}_W = \boldsymbol{p}_e - d_6 \boldsymbol{a}_e$$



- Compute wrist position

  $\boldsymbol{p}_W(q_1, q_2, q_3)$

- Solve inverse kinematics

  (q_1, q_2, q_3)

- Compute $\boldsymbol{R}_3^0(q_1, q_2, q_3)$
- Compute

  $\boldsymbol{R}_6^3(\vartheta_4, \vartheta_5, \vartheta_6) = \boldsymbol{R}_3^{0\,T} \boldsymbol{R}$

- Solve inverse kinematics $(\vartheta_4, \vartheta_5, \vartheta_6)$

Relationship between the joint velocities and the end-effector linear and angular velocities Jacobian

- Jacobian
  - Derivative of a rotation matrix
  - Jacobian computation
- Differential Kinematics
  - Kinematic singularities
  - Analysis of redundancy
  - Analytical Jacobian
- Inverse Kinematics Algorithms
  - Jacobian (pseudo-)inverse
  - Jacobian transpose
  - Orientation error

$$T_e(q) = \begin{bmatrix} R_e(q) & p_e(q) \\ 0^T & 1 \end{bmatrix}$$

- Differential kinematics equation

$$\dot{p}_e = J_P(q)\dot{q}$$

$$\omega_e = J_O(q)\dot{q}$$

$$v_e = \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = J(q)\dot{q}$$

$$J = \begin{bmatrix} J_P \\ J_O \end{bmatrix}$$

Derivative of a Rotation Matrix

$$\boldsymbol{R}(t)\boldsymbol{R}^T(t) = \boldsymbol{I}$$

- Differentiating

$$\dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) + \boldsymbol{R}(t)\dot{\boldsymbol{R}}^T(t) = \boldsymbol{O}$$

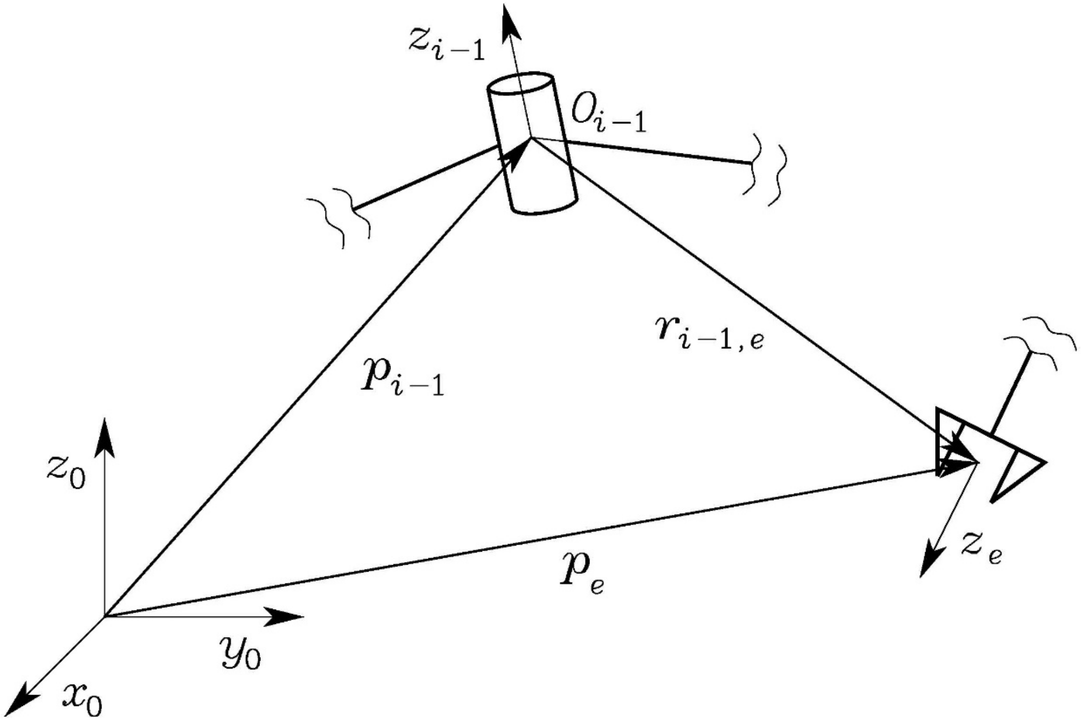
- Skew-symmetric operator

$$\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) \qquad \boldsymbol{S}(t) + \boldsymbol{S}^T(t) = \boldsymbol{O}$$

- Angular velocity

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{S}(\boldsymbol{\omega}(t))\boldsymbol{R}(t) \qquad \boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$\dot{\boldsymbol{p}}_e = \sum_{i=1}^{n} \frac{\partial \boldsymbol{p}_e}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Pi} \dot{q}_i$$



- Prismatic joint

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \dot{d}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1}$$

- Revolute joint

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,e} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})$$

$$\boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})$$

$$\boldsymbol{\omega}_e = \boldsymbol{\omega}_n = \sum_{i=1}^{n} \boldsymbol{\omega}_{i-1,i} = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Oi}\dot{q}_i$$

- Prismatic joint

$$\dot{q}_i\boldsymbol{\jmath}_{Oi} = \mathbf{0}$$
$$\boldsymbol{\jmath}_{Oi} = \mathbf{0}$$

- Revolute joint

$$\dot{q}_i\boldsymbol{\jmath}_{Oi} = \dot{\vartheta}_i\boldsymbol{z}_{i-1}$$
$$\boldsymbol{\jmath}_{Oi} = \boldsymbol{z}_{i-1}$$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{\jmath}_{P1} & & \boldsymbol{\jmath}_{Pn} \\ & \dots & \\ \boldsymbol{\jmath}_{O1} & & \boldsymbol{\jmath}_{On} \end{bmatrix}$$

- Prismatic joint

$$\begin{bmatrix} \boldsymbol{\jmath}_{Pi} \\ \boldsymbol{\jmath}_{Oi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{z}_{i-1} = \boldsymbol{R}_1^0(q_1) \dots \boldsymbol{R}_{i-1}^{i-2}(q_{i-1}) \boldsymbol{z}_0$$

- Revolute joint

$$\begin{bmatrix} \boldsymbol{\jmath}_{Pi} \\ \boldsymbol{\jmath}_{Oi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix}$$

$$\widetilde{\boldsymbol{p}}_e = \boldsymbol{A}_1^0(q_1) \dots \boldsymbol{A}_n^{n-1}(q_n) \widetilde{\boldsymbol{p}}_0$$

$$\widetilde{\boldsymbol{p}}_{i-1} = \boldsymbol{A}_1^0(q_1) \dots \boldsymbol{A}_{i-1}^{i-2}(q_{i-1}) \widetilde{\boldsymbol{p}}_0$$

$$v_e = J(q)\dot{q}$$

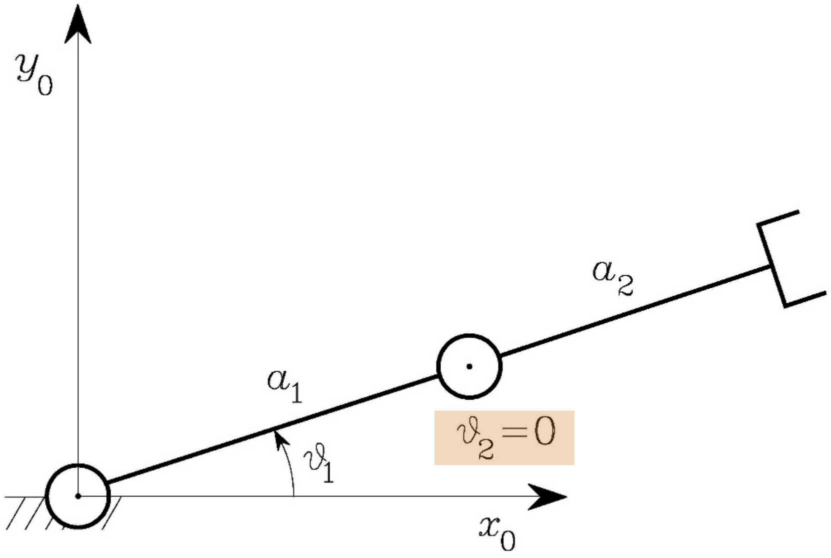
- Those configurations at which the Jacobian is rank-deficient are termed kinematic singularities
  - Reduced mobility (it is not possible to impose an arbitrary motion to the end-effector)
  - Infinite solutions to the inverse kinematics problem may exist
  - Small velocities in the operational space may cause large velocities in the joint space (In the neighbourhood of a singularity)
- Classification
  - Boundary singularities occurring when the manipulator is either outstretched or retracted (can be avoided)
  - Internal singularities occurring inside the reachable workspace and generally caused by the alignment of two or more axes of motion, or else by the attainment of particular end-effector configurations (can be encountered anywhere for a planned path in the operational space)

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\det(\boldsymbol{J}) = a_1 a_2 s_2$$

$$\Downarrow$$

$$\vartheta_2 = 0 \qquad \vartheta_2 = \pi$$



- The vectors $[-(a_1 + a_2)s_1 \quad (a_1 + a_2)c_1]^T$ and $[-a_2 s_1 \quad a_2 c_1]^T$ become parallel (tip velocity components are not independent)

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} \end{bmatrix}$$

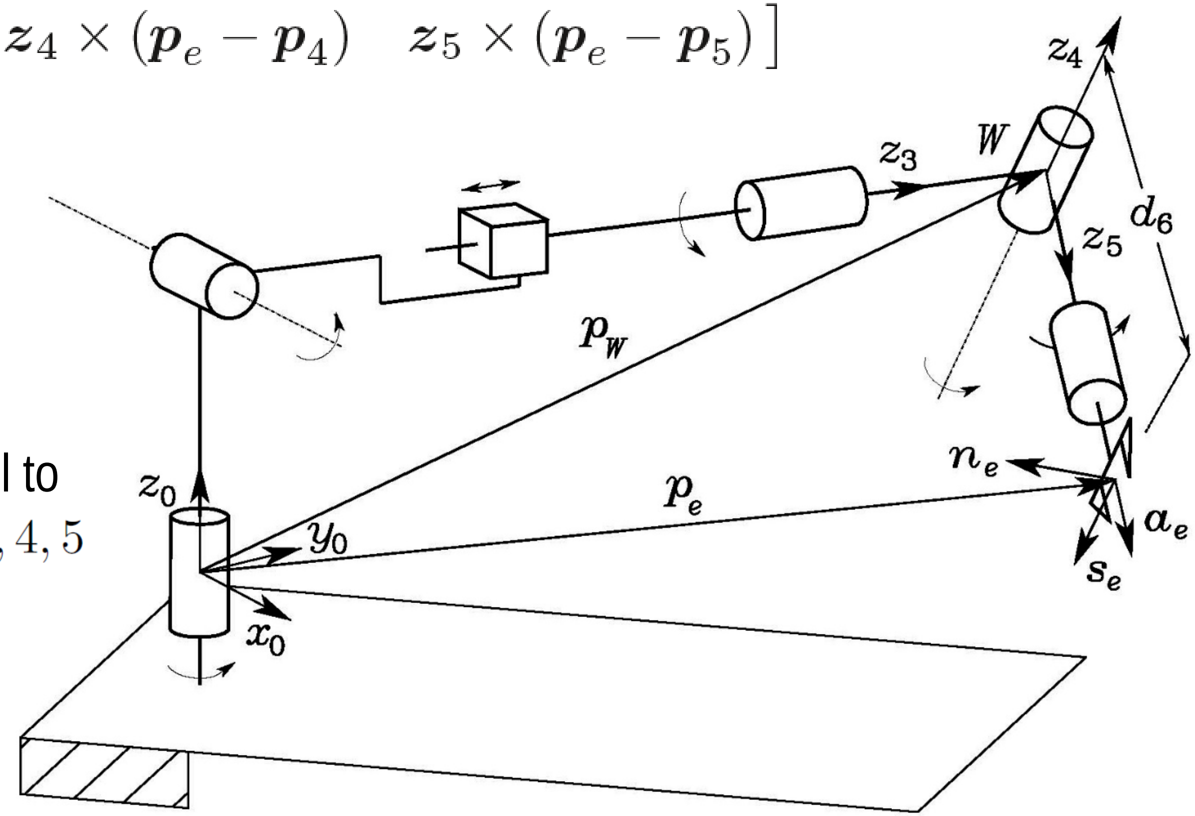
$$\boldsymbol{J}_{12} = \begin{bmatrix} \boldsymbol{z}_3 \times (\boldsymbol{p}_e - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p}_e - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p}_e - \boldsymbol{p}_5) \end{bmatrix}$$

$$\boldsymbol{J}_{22} = \begin{bmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$$

- Choosing $\boldsymbol{p}_e = \boldsymbol{p}_W$
  - Vectors $\boldsymbol{p}_W - \boldsymbol{p}_i$ parallel to the unit vectors $\boldsymbol{z}_i$, $i = 3, 4, 5$

$$\boldsymbol{J}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

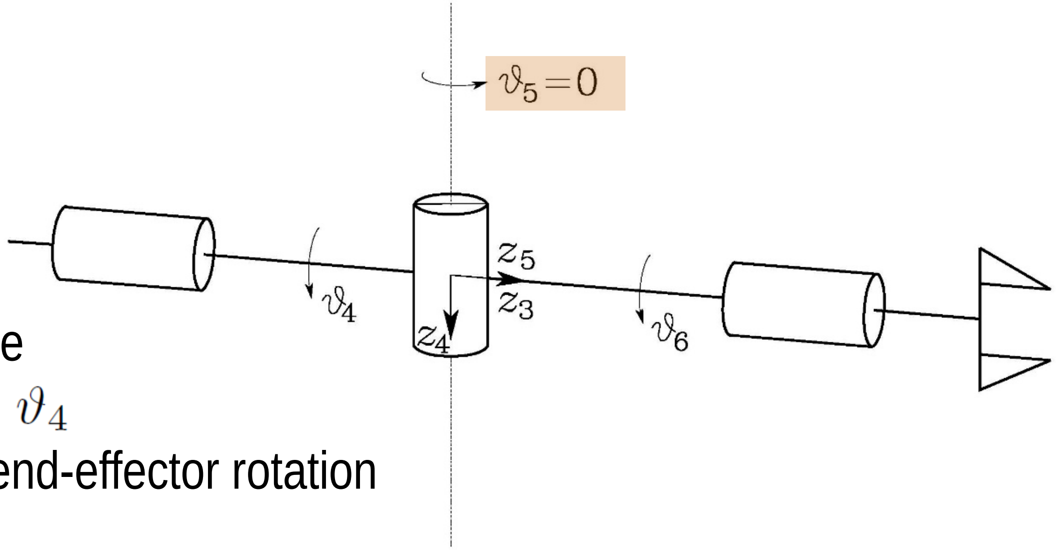
$$\det(\boldsymbol{J}) = \det(\boldsymbol{J}_{11})\det(\boldsymbol{J}_{22})$$



- $\boldsymbol{z}_3$ parallel to $\boldsymbol{z}_5$

  $\vartheta_5 = 0$ $\vartheta_5 = \pi$

  - Rotations of equal magnitude about opposite directions on ϑ_4 and ϑ_6 do not produce any end-effector rotation



Anthropomorphic Arm Singularities

$$\det(\boldsymbol{J}_P) = -a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

- Elbow singularity

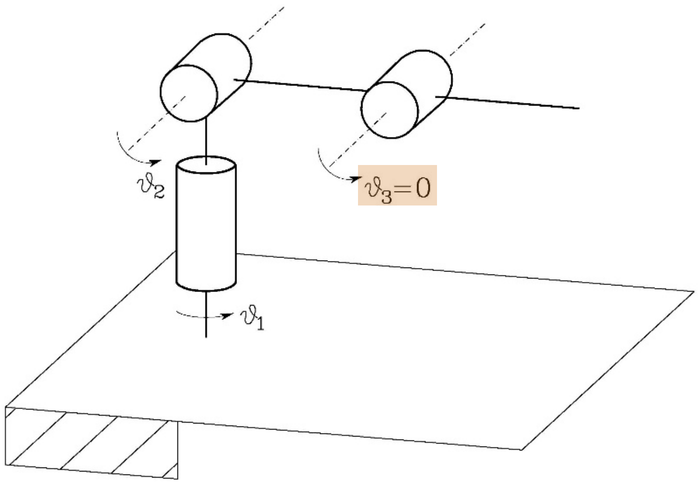
  $\vartheta_3 = 0 \qquad \vartheta_3 = \pi$

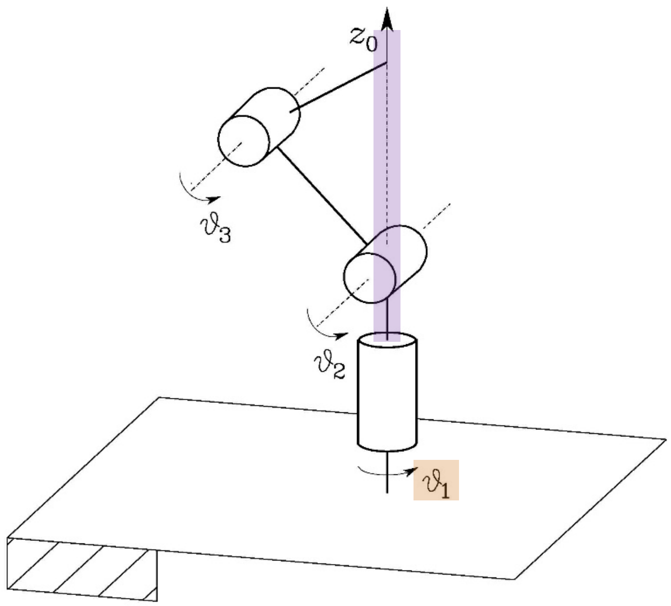
  - conceptually equivalent to the singularity found for the two-link planar arm

- Shoulder singularity

  $p_x = p_y = 0$

  - A rotation of ϑ_1 does not cause any translation of the wrist position





Analysis of Redundancy

- Differential kinematics

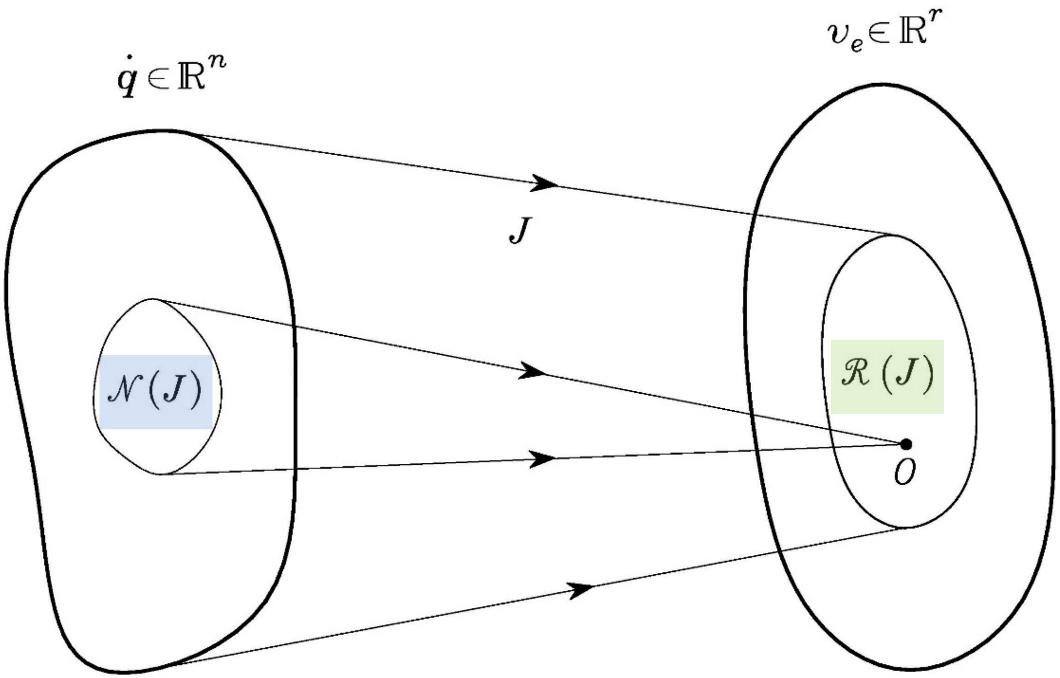
$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

  - If $\varrho(\boldsymbol{J}) = r$

$$\dim(\mathcal{R}(\boldsymbol{J})) = r$$
$$\dim(\mathcal{N}(\boldsymbol{J})) = n - r$$

  - In general

$$\dim(\mathcal{R}(\boldsymbol{J})) + \dim(\mathcal{N}(\boldsymbol{J})) = n$$



- If $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_a \qquad \mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{J})$$

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}^* + \boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}_e$$

  - $\dot{\boldsymbol{q}}_0$ generates internal motions of the structure

Inversion of Differential Kinematics Equation

- Nonlinear kinematics equation between the joint space and the operational space
- Differential kinematics equation represents a linear mapping between the joint velocity space and the operational velocity space
- Given an end-effector velocity $\boldsymbol{v}_e$ + initial conditions, compute a feasible joint trajectory $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ that reproduces the given trajectory
  - If $n = r$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}$$

$$\boldsymbol{q}(t) = \int_0^t \dot{\boldsymbol{q}}(\varsigma)d\varsigma + \boldsymbol{q}(0)$$

  - Numerical integration rule (Euler) $\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \dot{\boldsymbol{q}}(t_k)\Delta t$

- Local optimal solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v}_e + (\boldsymbol{I}_n - \boldsymbol{J}^{\dagger}\boldsymbol{J})\dot{\boldsymbol{q}}_0$$

- Internal motions

$$\dot{\boldsymbol{q}}_0 = k_0 \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T$$

  - Manipulability measure $w(\boldsymbol{q}) = \sqrt{\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)}$
  - Distance from mechanical joint limits $w(\boldsymbol{q}) = -\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}}\right)^2$
  - Distance from an obstacle $w(\boldsymbol{q}) = \min_{\boldsymbol{p},\boldsymbol{o}} \|\boldsymbol{p}(\boldsymbol{q}) - \boldsymbol{o}\|$

- The above solutions can be computed only when the Jacobian has full rank
- Whenever $\boldsymbol{J}$ is not full rank
  - If $\boldsymbol{v}_e \in \mathcal{R}(\boldsymbol{J}) \Longrightarrow$ It is possible to find a solution $\dot{\boldsymbol{q}}$ by extracting all the linearly independent equations (assigned path physically executable by the manipulator)
  - If $\boldsymbol{v}_e \notin \mathcal{R}(\boldsymbol{J}) \Longrightarrow$ The system of equations has no solution (non executable path at manipulator's given posture)
- Inversion in the neighborhood of singularities: Damped least-squares (DLS) inverse

$$\boldsymbol{J}^{\star} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + k^2\boldsymbol{I})^{-1}$$

$$\dot{\boldsymbol{p}}_e = \frac{\partial \boldsymbol{p}_e}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{\phi}}_e = \frac{\partial \boldsymbol{\phi}_e}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_\phi(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \dot{\boldsymbol{\phi}}_e \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q}) \end{bmatrix}\dot{\boldsymbol{q}} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$$

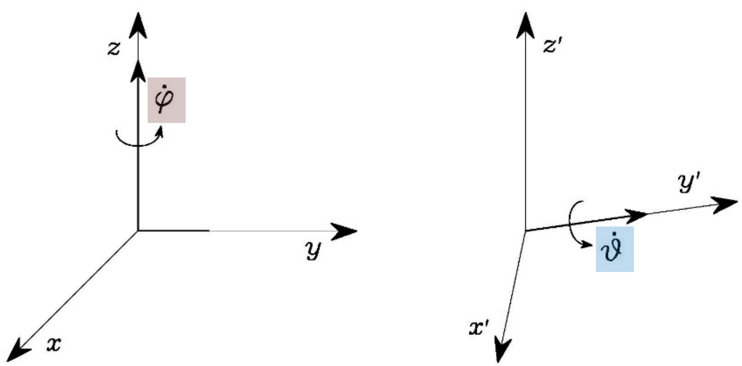
- Analytical Jacobian

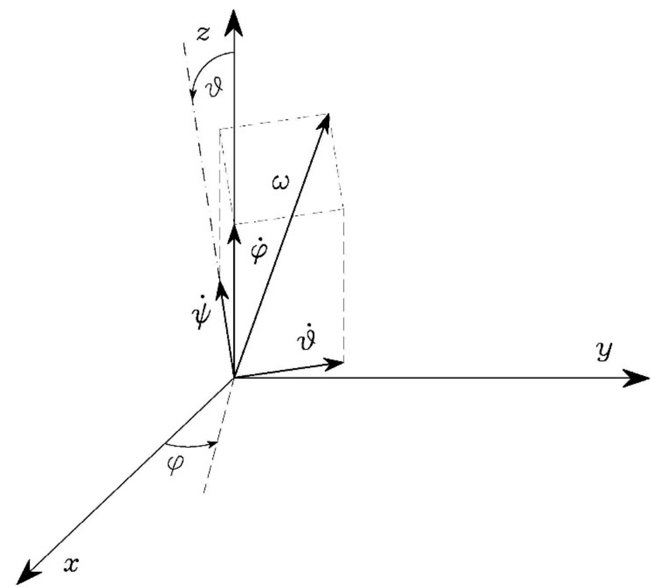
$$\boldsymbol{J}_A(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

  - $\boldsymbol{\phi}_e(\boldsymbol{q})$ is not usually available in direct form, but requires computation of the elements of the relative rotation matrix

Angular Velocity and Derivatives of Euler Angles

- Rotational velocities of Euler angles ZYZ in current frame
  - As a result of $\dot{\varphi}$: $\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T = \dot{\varphi}\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$
  - As a result of $\dot{\vartheta}$: $\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T = \dot{\vartheta}\begin{bmatrix}-s_\varphi & c_\varphi & 0\end{bmatrix}^T$
  - As a result of $\dot{\psi}$: $\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}^T = \dot{\psi}\begin{bmatrix}c_\varphi s_\vartheta & s_\varphi s_\vartheta & c_\vartheta\end{bmatrix}^T$





- Composition of elementary rotational velocities

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix} \dot{\boldsymbol{\phi}} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$

  - Representation singularity for $\vartheta = 0, \pi$

Relationship Between Jacobians

$$\boldsymbol{v}_e = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}(\boldsymbol{\phi}_e) \end{bmatrix} \dot{\boldsymbol{x}}_e = \boldsymbol{T}_A(\boldsymbol{\phi}_e)\dot{\boldsymbol{x}}_e$$

$$\boldsymbol{J} = \boldsymbol{T}_A(\boldsymbol{\phi})\boldsymbol{J}_A$$

- Geometric Jacobian
  - Quantities of clear physical meaning
- Analytical Jacobian
  - Differential quantities of variables defined in the operational space

Composition of elementary rotational velocities

Closed-Loop Inverse Kinematics Schemes

- Algorithmic solution

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \boldsymbol{J}^{-1}(\boldsymbol{q}(t_k))\boldsymbol{v}_e(t_k)\Delta t$$

  - Solution drift

- Operational space error

$$\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}_e$$

- Differentiating …

$$\begin{aligned}\dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}}_e \\ &= \dot{\boldsymbol{x}}_d - \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned}$$

- Find $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{e})$: $\quad \boldsymbol{e} \to \boldsymbol{0}$

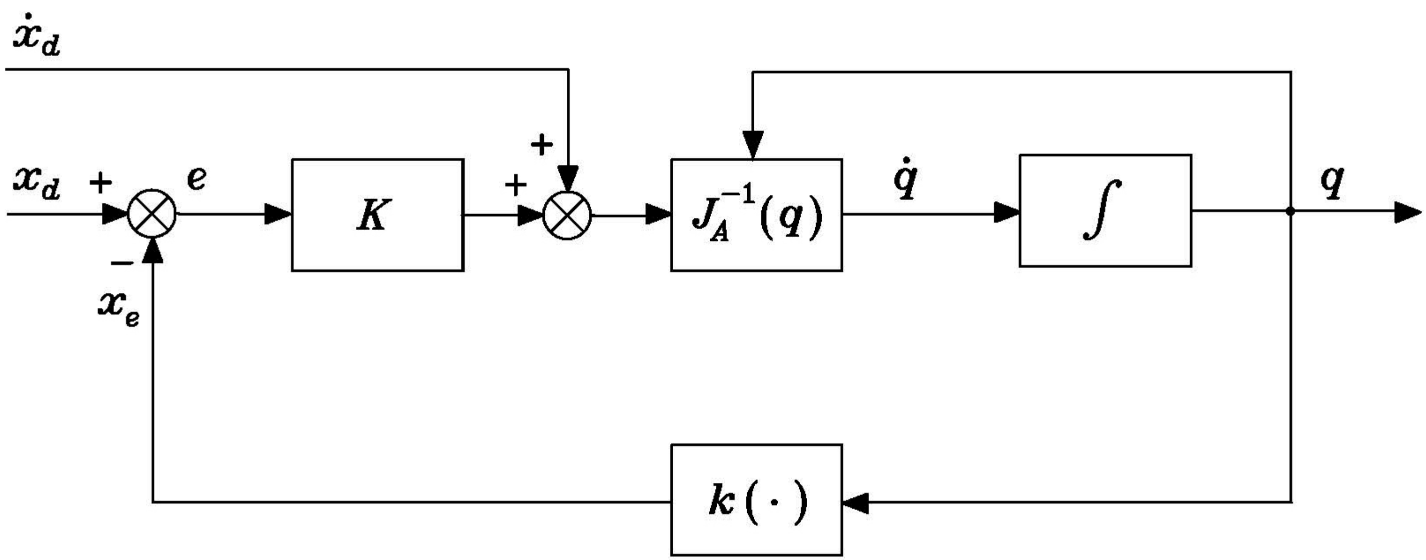
Jacobian Pseudo-Inverse CLIK Scheme

- Error dynamics linearization

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q})(\dot{\boldsymbol{x}}_d + \boldsymbol{K}\boldsymbol{e}) \quad \Longrightarrow \quad \dot{\boldsymbol{e}} + \boldsymbol{K}\boldsymbol{e} = \boldsymbol{0}$$

- For a redundant manipulator

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{\dagger}(\dot{\boldsymbol{x}}_d + \boldsymbol{K}\boldsymbol{e}) + (\boldsymbol{I}_n - \boldsymbol{J}_A^{\dagger}\boldsymbol{J}_A)\dot{\boldsymbol{q}}_0$$



Jacobian Transpose CLIK Scheme

- $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{e})$ without linearizing error dynamics

Lyapunov method

$$V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{K}\boldsymbol{e} \qquad V(\boldsymbol{e}) > 0 \quad \forall \boldsymbol{e} \neq \boldsymbol{0} \qquad V(\boldsymbol{0}) = 0$$

- Differentiating ... $\dot{V} = \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_e$

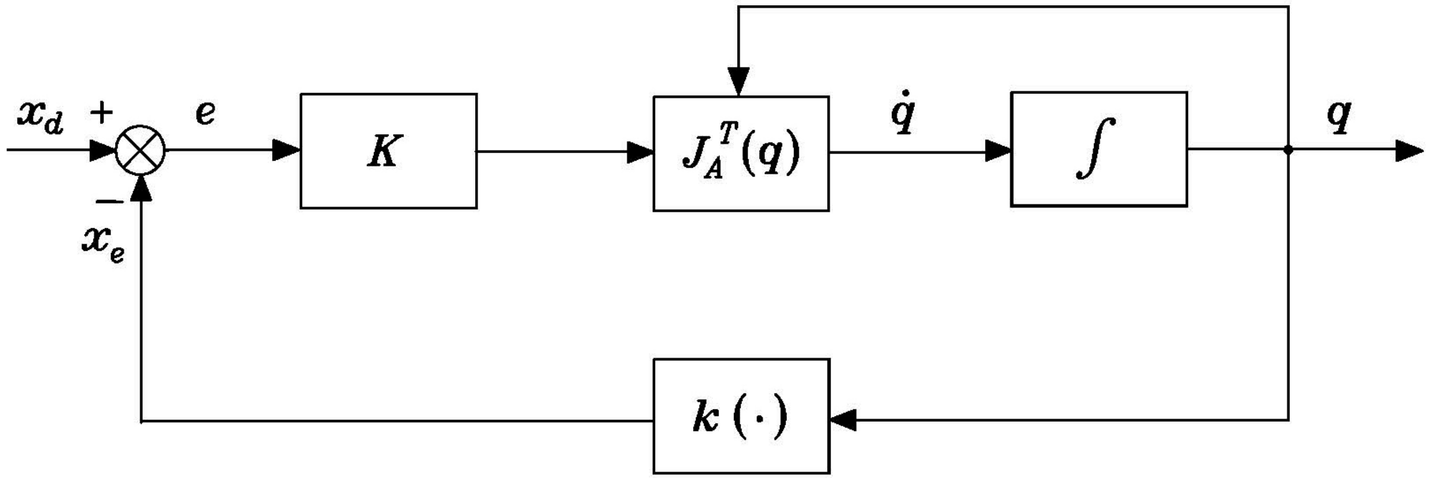
$$= \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- Choosing $\dot{\boldsymbol{q}} = \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e} \implies \dot{V} = \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\boldsymbol{J}_A(\boldsymbol{q})\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}$
  - If $\dot{\boldsymbol{x}}_d = \boldsymbol{0} \implies \dot{V} < 0$ with $V > 0$ (asymptotic stability)
  - If $\mathcal{N}(\boldsymbol{J}_A^T) \neq \emptyset \implies \dot{V} = 0$ if $\boldsymbol{K}\boldsymbol{e} \in \mathcal{N}(\boldsymbol{J}_A^T)$

    $\dot{\boldsymbol{q}} = \boldsymbol{0}$ with $\boldsymbol{e} \neq \boldsymbol{0}$ (stuck?)

Jacobian Transpose CLIK Scheme[2]

- If $\dot{\boldsymbol{x}}_d \neq \mathbf{0}$
  - $\boldsymbol{e}(t)$ bounded (increase norm of $\boldsymbol{K}$)
  - $\boldsymbol{e}(\infty) \rightarrow \mathbf{0}$

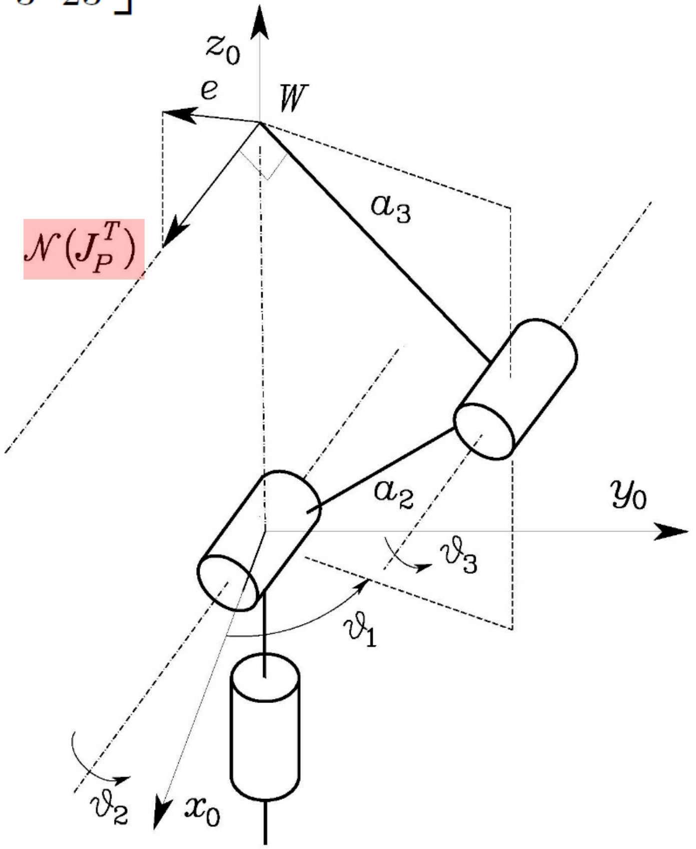


$$J_P^T = \begin{bmatrix} 0 & 0 & 0 \\ -c_1(a_2 s_2 + a_3 s_{23}) & -s_1(a_2 s_2 + a_3 s_{23}) & 0 \\ -a_3 c_1 s_{23} & -a_3 s_1 s_{23} & a_3 c_{23} \end{bmatrix}$$

- Null space (shoulder singularity)

$$\frac{\nu_y}{\nu_x} = -\frac{1}{\tan\vartheta_1} \qquad \nu_z = 0$$

  - If desired path is along the line normal to the plane of the structure at the intersection with the wrist point $\Longrightarrow$ algorithm gets stuck (end-effector cannot move)
  - If desired path has a non-null component in the plane of the structure $\Longrightarrow$ algorithm convergence is ensured



Orientation Error

- Position error

$$\boldsymbol{e}_P = \boldsymbol{p}_d - \boldsymbol{p}_e(\boldsymbol{q})$$

$$\dot{\boldsymbol{e}}_P = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}_e$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q}) \begin{bmatrix} \dot{\boldsymbol{p}}_d + \boldsymbol{K}_P \boldsymbol{e}_P \\ \dot{\boldsymbol{\phi}}_d + \boldsymbol{K}_O \boldsymbol{e}_O \end{bmatrix}$$

- Orientation error

$$\boldsymbol{e}_O = \boldsymbol{\phi}_d - \boldsymbol{\phi}_e(\boldsymbol{q})$$

$$\dot{\boldsymbol{e}}_O = \dot{\boldsymbol{\phi}}_d - \dot{\boldsymbol{\phi}}_e$$

- Easy to specify $\boldsymbol{\phi}_d(t)$
- Requires computation of $\boldsymbol{\phi}_e$ with inverse formulae from $\boldsymbol{R}_e = [\boldsymbol{n}_e \quad \boldsymbol{s}_e \quad \boldsymbol{a}_e]$
- Manipulator with spherical wrist
  - Compute $\boldsymbol{q}_P \Longrightarrow \boldsymbol{R}_W$
  - Compute $\boldsymbol{R}_W^T \boldsymbol{R}_d \Longrightarrow \boldsymbol{q}_O$ (ZYZ Euler angles)

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_d \boldsymbol{R}_e^T(\boldsymbol{q})$$

- Orientation error

$$\boldsymbol{e}_O = \boldsymbol{r} \sin\vartheta \qquad -\pi/2 < \vartheta < \pi/2$$

$$= \frac{1}{2}(\boldsymbol{n}_e(\boldsymbol{q}) \times \boldsymbol{n}_d + \boldsymbol{s}_e(\boldsymbol{q}) \times \boldsymbol{s}_d + \boldsymbol{a}_e(\boldsymbol{q}) \times \boldsymbol{a}_d)$$

$$\boldsymbol{n}_e^T \boldsymbol{n}_d \geq 0 \qquad \boldsymbol{s}_e^T \boldsymbol{s}_d \geq 0 \qquad \boldsymbol{a}_e^T \boldsymbol{a}_d \geq 0$$

- Differentiating …

$$\dot{\boldsymbol{e}}_O = \boldsymbol{L}^T \boldsymbol{\omega}_d - \boldsymbol{L}\boldsymbol{\omega}_e \quad \boldsymbol{L} = -\frac{1}{2}\left(\boldsymbol{S}(\boldsymbol{n}_d)\boldsymbol{S}(\boldsymbol{n}_e) + \boldsymbol{S}(\boldsymbol{s}_d)\boldsymbol{S}(\boldsymbol{s}_e) + \boldsymbol{S}(\boldsymbol{a}_d)\boldsymbol{S}(\boldsymbol{a}_e)\right)$$

$$\dot{\boldsymbol{e}} = \begin{bmatrix} \dot{\boldsymbol{e}}_P \\ \dot{\boldsymbol{e}}_O \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{p}}_d - \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}} \\ \boldsymbol{L}^T\boldsymbol{\omega}_d - \boldsymbol{L}\boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{p}}_d \\ \boldsymbol{L}^T\boldsymbol{\omega}_d \end{bmatrix} - \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L} \end{bmatrix} \boldsymbol{J}\dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{bmatrix} \dot{\boldsymbol{p}}_d + \boldsymbol{K}_P \boldsymbol{e}_P \\ \boldsymbol{L}^{-1}\left(\boldsymbol{L}^T\boldsymbol{\omega}_d + \boldsymbol{K}_O \boldsymbol{e}_O\right) \end{bmatrix}$$

$$\Delta\mathcal{Q} = \mathcal{Q}_d * \mathcal{Q}_e^{-1}$$

- Orientation error

$$\boldsymbol{e}_O = \Delta\boldsymbol{\epsilon} = \eta_e(\boldsymbol{q})\boldsymbol{\epsilon}_d - \eta_d\boldsymbol{\epsilon}_e(\boldsymbol{q}) - \boldsymbol{S}(\boldsymbol{\epsilon}_d)\boldsymbol{\epsilon}_e(\boldsymbol{q})$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \begin{bmatrix} \dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\boldsymbol{e}_P \\ \boldsymbol{\omega}_d + \boldsymbol{K}_O\boldsymbol{e}_O \end{bmatrix} \Longrightarrow \boldsymbol{\omega}_d - \boldsymbol{\omega} + \boldsymbol{K}_O\boldsymbol{e}_O = \mathbf{0}$$

- Quaternion propagation

$$\dot{\eta}_e = -\frac{1}{2}\boldsymbol{\epsilon}_e^T\boldsymbol{\omega}_e$$

$$\dot{\boldsymbol{\epsilon}}_e = \frac{1}{2}\left(\eta_e\boldsymbol{I}_3 - \boldsymbol{S}(\boldsymbol{\epsilon}_e)\right)\boldsymbol{\omega}_e$$

- Stability analysis

$$V = (\eta_d - \eta_e)^2 + (\boldsymbol{\epsilon}_d - \boldsymbol{\epsilon}_e)^T(\boldsymbol{\epsilon}_d - \boldsymbol{\epsilon}_e) \qquad \dot{V} = -\boldsymbol{e}_O^T\boldsymbol{K}_O\boldsymbol{e}_O$$

Second-order Inverse Kinematics Algorithms

$$\dot{\boldsymbol{x}}_e = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$$

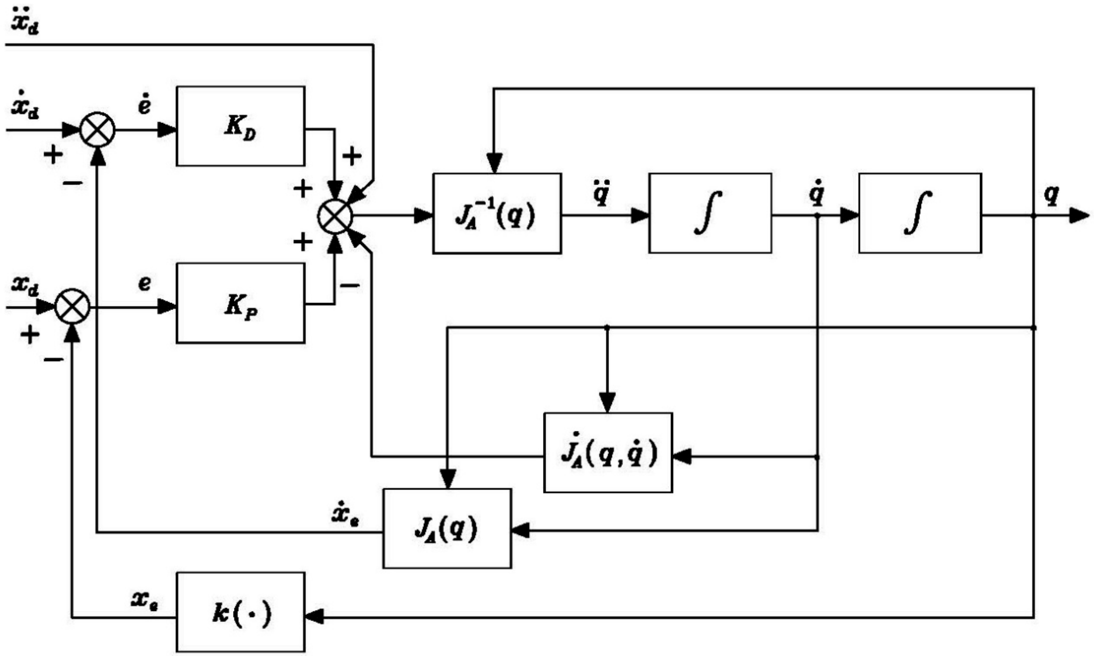
- Differentiating …

$$\ddot{\boldsymbol{x}}_e = \boldsymbol{J}_A(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_A(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

- Error dynamics

$$\ddot{\boldsymbol{e}} = \ddot{\boldsymbol{x}}_d - \ddot{\boldsymbol{x}}_e$$
$$= \ddot{\boldsymbol{x}}_d - \boldsymbol{J}_A(\boldsymbol{q})\ddot{\boldsymbol{q}} - \dot{\boldsymbol{J}}_A(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q})\left(\ddot{\boldsymbol{x}}_e - \dot{\boldsymbol{J}}_A(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right) \Longrightarrow \ddot{\boldsymbol{e}} + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e} = \boldsymbol{0}$$



Relationship between the generalized forces applied to the end-effector (forces) and the generalized forces applied to the joints (torques), with the manipulator at an equilibrium configuration

- Elementary work associated with joint torques $dW_{\tau} = \boldsymbol{\tau}^T d\boldsymbol{q}$
- Elementary work associated with end-effector forces

$$
\begin{aligned}
dW_{\gamma} &= \boldsymbol{f}_e^T d\boldsymbol{p}_e + \boldsymbol{\mu}_e^T \boldsymbol{\omega}_e dt \\
&= \boldsymbol{f}_e^T \boldsymbol{J}_P(\boldsymbol{q}) d\boldsymbol{q} + \boldsymbol{\mu}_e^T \boldsymbol{J}_O(\boldsymbol{q}) d\boldsymbol{q} \\
&= \boldsymbol{\gamma}_e^T \boldsymbol{J}(\boldsymbol{q}) d\boldsymbol{q}
\end{aligned}
$$

- Elementary displacements $\equiv$ virtual displacements

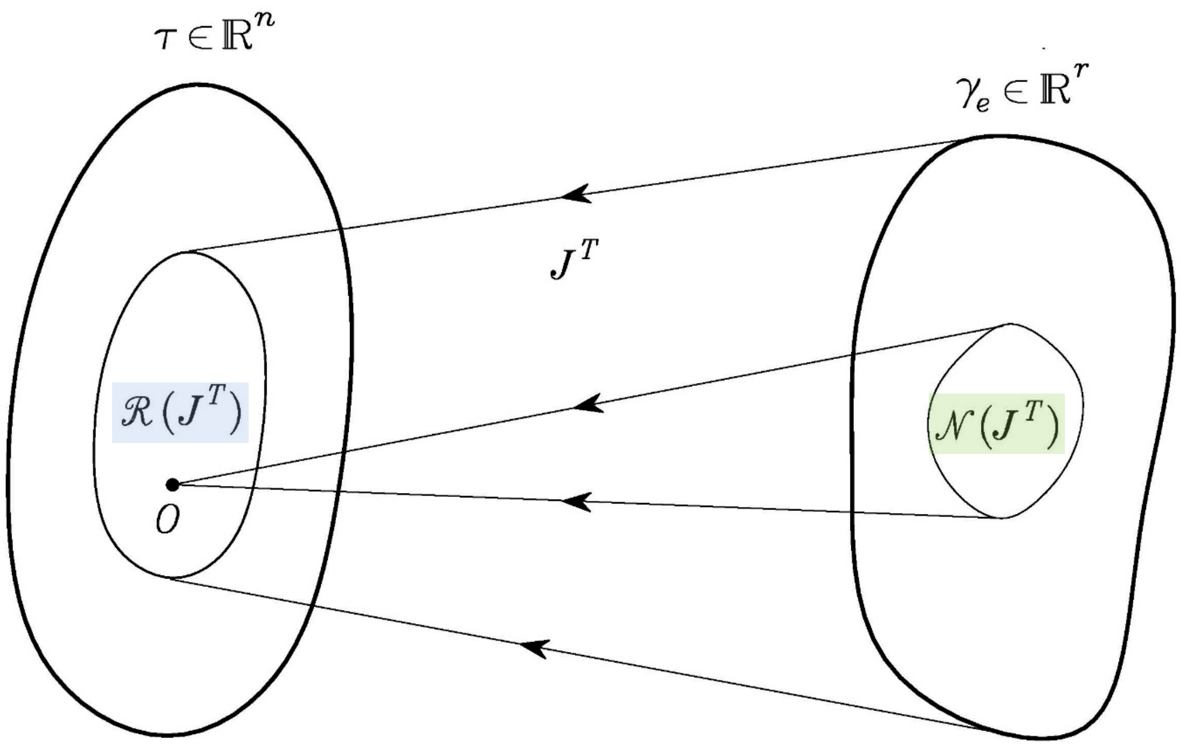
$$
\delta W_{\tau} = \boldsymbol{\tau}^T \delta\boldsymbol{q} \qquad \delta W_{\gamma} = \boldsymbol{\gamma}_e^T \boldsymbol{J}(\boldsymbol{q}) \delta\boldsymbol{q}
$$

- Principle of virtual work: the manipulator is at static equilibrium if and only if

$$
\delta W_{\tau} = \delta W_{\gamma} \qquad \forall \delta\boldsymbol{q} \qquad \Longrightarrow \qquad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}) \boldsymbol{\gamma}_e
$$

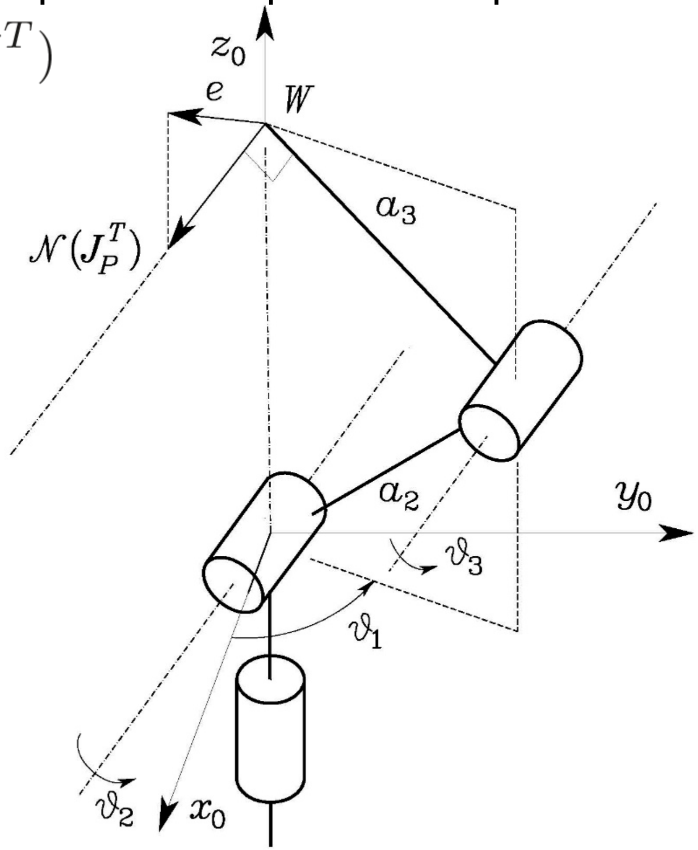
$\mathcal{N}(\boldsymbol{J}) \equiv \mathcal{R}^{\perp}(\boldsymbol{J}^T)$ $\mathcal{R}(\boldsymbol{J}) \equiv \mathcal{N}^{\perp}(\boldsymbol{J}^T)$

- End-effector forces $\gamma_e \in \mathcal{N}(\boldsymbol{J}^T)$ not requiring any balancing joint torques, in the given manipulator posture



Jacobian Transpose CLIK Scheme

- Physical interpretation of CLIK scheme with Jacobian transpose
  - Ideal dynamics $\boldsymbol{\tau} = \dot{\boldsymbol{q}}$ (null masses and unit viscous friction coefficients)
  - Elastic force $\boldsymbol{K}\boldsymbol{e}$ pulling end-effector towards desired posture in operational space
  - Manipulator is allowed to move only if $\boldsymbol{K}\boldsymbol{e} \notin \mathcal{N}(\boldsymbol{J}^T)$



Relationship between the joint actuator torques and the motion of the structure

- Lagrangian Formulation
  - Equations of motion
  - Notable properties of dynamic model
- Direct dynamics and inverse dynamics

Lagrange Formulation

- Lagrangian = Kinetic energy – Potential energy

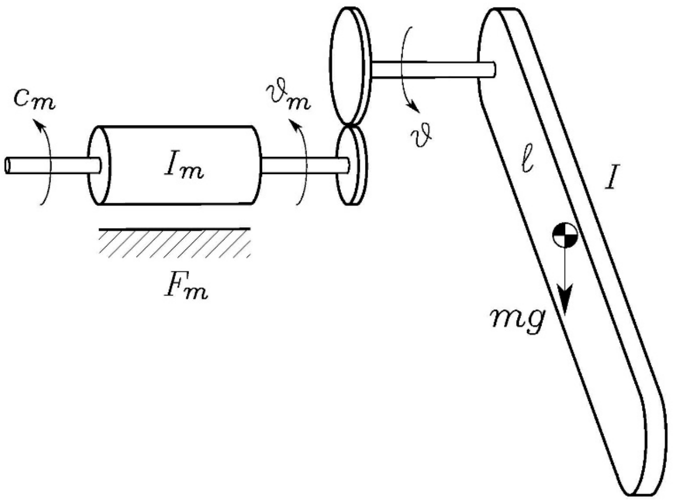
$$\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q})$$

- Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{\xi}$$

  - $\boldsymbol{\xi}$: generalized forces associated with generalized coordinates $\boldsymbol{q}$

Dynamic Model of Pendulum



- Kinetic energy

$$\mathcal{T} = \frac{1}{2} I \dot{\vartheta}^2 + \frac{1}{2} I_m k_r^2 \dot{\vartheta}^2$$

- Potential energy

$$\mathcal{U} = mg\ell(1 - \cos\vartheta)$$

- Lagrangian

$$\mathcal{L} = \frac{1}{2} I \dot{\vartheta}^2 + \frac{1}{2} I_m k_r^2 \dot{\vartheta}^2 - mg\ell(1 - \cos\vartheta)$$

- Equations of motion

$$(I + I_m k_r^2)\ddot{\vartheta} + mg\ell \sin\vartheta = \xi \implies (I + I_m k_r^2)\ddot{\vartheta} + (F + F_m k_r^2)\dot{\vartheta} + mg\ell \sin\vartheta = \tau$$

Kinetic Energy and Potential Energy

- Contributions relative to the motion of each link and each joint actuator

$$\mathcal{T} = \sum_{i=1}^{n} (\mathcal{T}_{\ell_i} + \mathcal{T}_{m_i}) \qquad \mathcal{U} = \sum_{i=1}^{n} (\mathcal{U}_{\ell_i} + \mathcal{U}_{m_i})$$

- Lagrangian

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q})$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} \left(m_{\ell_i} \boldsymbol{g}_0^T \boldsymbol{p}_{\ell_i}(\boldsymbol{q}) + m_{m_i} \boldsymbol{g}_0^T \boldsymbol{p}_{m_i}(\boldsymbol{q}) \right)$$

$$\frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

- Inertia matrix
  - symmetric
  - positive definite
  - configuration-dependent

$$\boldsymbol{B}(\boldsymbol{q}) = \sum_{i=1}^{n} \Big(m_{\ell_i} \boldsymbol{J}_P^{(\ell_i)T} \boldsymbol{J}_P^{(\ell_i)} + \boldsymbol{J}_O^{(\ell_i)T} \boldsymbol{R}_i \boldsymbol{I}_{\ell_i}^{i} \boldsymbol{R}_i^T \boldsymbol{J}_O^{(\ell_i)}$$

$$+ m_{m_i} \boldsymbol{J}_P^{(m_i)T} \boldsymbol{J}_P^{(m_i)} + \boldsymbol{J}_O^{(m_i)T} \boldsymbol{R}_{m_i} \boldsymbol{I}_{m_i}^{m_i} \boldsymbol{R}_{m_i}^T \boldsymbol{J}_O^{(m_i)} \Big)$$

- Taking various derivatives …

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\xi}$$

$$\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T + \left(\frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^{n}\frac{db_{ij}(\boldsymbol{q})}{dt}\dot{q}_j \qquad \frac{\partial \mathcal{T}}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{jk}(\boldsymbol{q})}{\partial q_i}\dot{q}_k\dot{q}_j$$

$$= \sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{ij}(\boldsymbol{q})}{\partial q_k}\dot{q}_k\dot{q}_j$$

$$\frac{\partial \mathcal{U}}{\partial q_i} = -\sum_{j=1}^{n}\left(m_{\ell_j}\boldsymbol{g}_0^T\frac{\partial \boldsymbol{p}_{\ell_j}}{\partial q_i} + m_{m_j}\boldsymbol{g}_0^T\frac{\partial \boldsymbol{p}_{m_j}}{\partial q_i}\right)$$

$$= -\sum_{j=1}^{n}\left(m_{\ell_j}\boldsymbol{g}_0^T\boldsymbol{j}_{Pi}^{(\ell_j)}(\boldsymbol{q}) + m_{m_j}\boldsymbol{g}_0^T\boldsymbol{j}_{Pi}^{(m_j)}(\boldsymbol{q})\right) = g_i(\boldsymbol{q})$$

Physical Interpretation

$$\sum_{j=1}^{n} b_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(\boldsymbol{q})\dot{q}_k\dot{q}_j + g_i(\boldsymbol{q}) = \xi_i \qquad i = 1, \dots, n$$

$$h_{ijk} = \frac{\partial b_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial b_{jk}}{\partial q_i}$$

- Acceleration terms
  - The coefficient b_{ii} represents the moment of inertia at Joint i axis, in the current manipulator posture, when the other joints are blocked
  - The coefficient b_{ij} accounts for the effect of acceleration of Joint j on Joint i
- Quadratic velocity terms
  - The term $h_{ijj}\dot{q}_j^2$ represents the centrifugal effect induced on Joint i by velocity of Joint j
  - $h_{iii} = 0$ $\partial b_{ii}/\partial q_i = 0$ ...s the Coriolis effect induced on Joint i by velocities of Joints j and k $h_{ijk}\dot{q}_j\dot{q}_k$
- Configuration-dependent term (gravity)
  - The term g_i represents the torque at Joint i axis of the manipulator in the current posture

Joint Space Dynamic Model

- Nonconservative forces doing work at manipulator joints
  - Actuation torques $\boldsymbol{\tau}$
  - Viscous friction torques $\boldsymbol{F}_v \dot{\boldsymbol{q}}$
  - Static friction torques (Coulomb model) $\boldsymbol{F}_s \,\mathbf{sgn}\,(\dot{\boldsymbol{q}})$
  - Balancing torques induced at joints by contact forces $\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$

- Equations of motion

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}_v\dot{\boldsymbol{q}} + \boldsymbol{F}_s\,\mathbf{sgn}\,(\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$$

  - $\boldsymbol{C}$: suitable $(n \times n)$ matrix so that $\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_k\dot{q}_j$

Skew-Symmetry of Matrix

- Elements of $\boldsymbol{C}$

$$c_{ij} = \sum_{k=1}^{n} c_{ijk}\dot{q}_k$$

  - Christoffel symbols of first type $c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$

- Notable property

$$\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\boldsymbol{N}^T(\boldsymbol{q},\dot{\boldsymbol{q}})$$

$$\boldsymbol{w}^T\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{w} = 0 \qquad \forall\boldsymbol{w}$$

  - If $\boldsymbol{w} = \dot{\boldsymbol{q}}$

$$\dot{\boldsymbol{q}}^T\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = 0 \qquad \forall\boldsymbol{C}$$

    principle of conservation of energy (Hamilton)

Linearity in the Dynamic Parameters

- Dynamic parameters
  - Mass of link and of motor (augmented link)
  - First inertia moment of augmented link
  - Inertia tensor of augmented link
  - Moment of inertia of rotor

$$\boldsymbol{\pi}_i = [m_i \quad m_i\ell_{C_ix} \quad m_i\ell_{C_iy} \quad m_i\ell_{C_iz} \quad \widehat{I}_{ixx} \quad \widehat{I}_{ixy} \quad \widehat{I}_{ixz} \quad \widehat{I}_{iyy} \quad \widehat{I}_{iyz} \quad \widehat{I}_{izz} \quad I_{m_i}]^T$$

- Both kinetic energy and potential energy are linear in the parameters

$$\mathcal{L} = \sum_{i=1}^{n} (\boldsymbol{\beta}_{\mathcal{T}i}^T - \boldsymbol{\beta}_{\mathcal{U}i}^T)\boldsymbol{\pi}_i$$

- Notable property

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{11}^T & \boldsymbol{y}_{12}^T & \dots & \boldsymbol{y}_{1n}^T \\ \boldsymbol{0}^T & \boldsymbol{y}_{22}^T & \dots & \boldsymbol{y}_{2n}^T \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0}^T & \boldsymbol{0}^T & \dots & \boldsymbol{y}_{nn}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \\ \vdots \\ \boldsymbol{\pi}_n \end{bmatrix} \qquad \boldsymbol{\tau} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\pi}$$

Direct Dynamics and Inverse Dynamics

- Direct dynamics (useful for simulation)
  - Given $\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0), \boldsymbol{\tau}(t)$ (and $\boldsymbol{h}_e(t)$), compute $\ddot{\boldsymbol{q}}(t), \dot{\boldsymbol{q}}(t), \boldsymbol{q}(t)$ for $t > t_0$

$$\ddot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})(\boldsymbol{\tau} - \boldsymbol{\tau}')$$

$$\boldsymbol{\tau}'(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}_v\dot{\boldsymbol{q}} + \boldsymbol{F}_s \,\mathbf{sgn}\,(\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$$

  - Given $\boldsymbol{q}(t_k)$, $\dot{\boldsymbol{q}}(t_k)$, $\boldsymbol{\tau}(t_k)$, compute $\ddot{\boldsymbol{q}}(t_k)$ and and numerically integrate with step Δt : $\dot{\boldsymbol{q}}(t_{k+1})$, $\boldsymbol{q}(t_{k+1})$

- Inverse dynamics (useful for planning and control)
  - Given $\ddot{\boldsymbol{q}}(t), \dot{\boldsymbol{q}}(t), \boldsymbol{q}(t)$ (and $\boldsymbol{h}_e(t)$) compute $\boldsymbol{\tau}(t)$

Robot Modelling

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