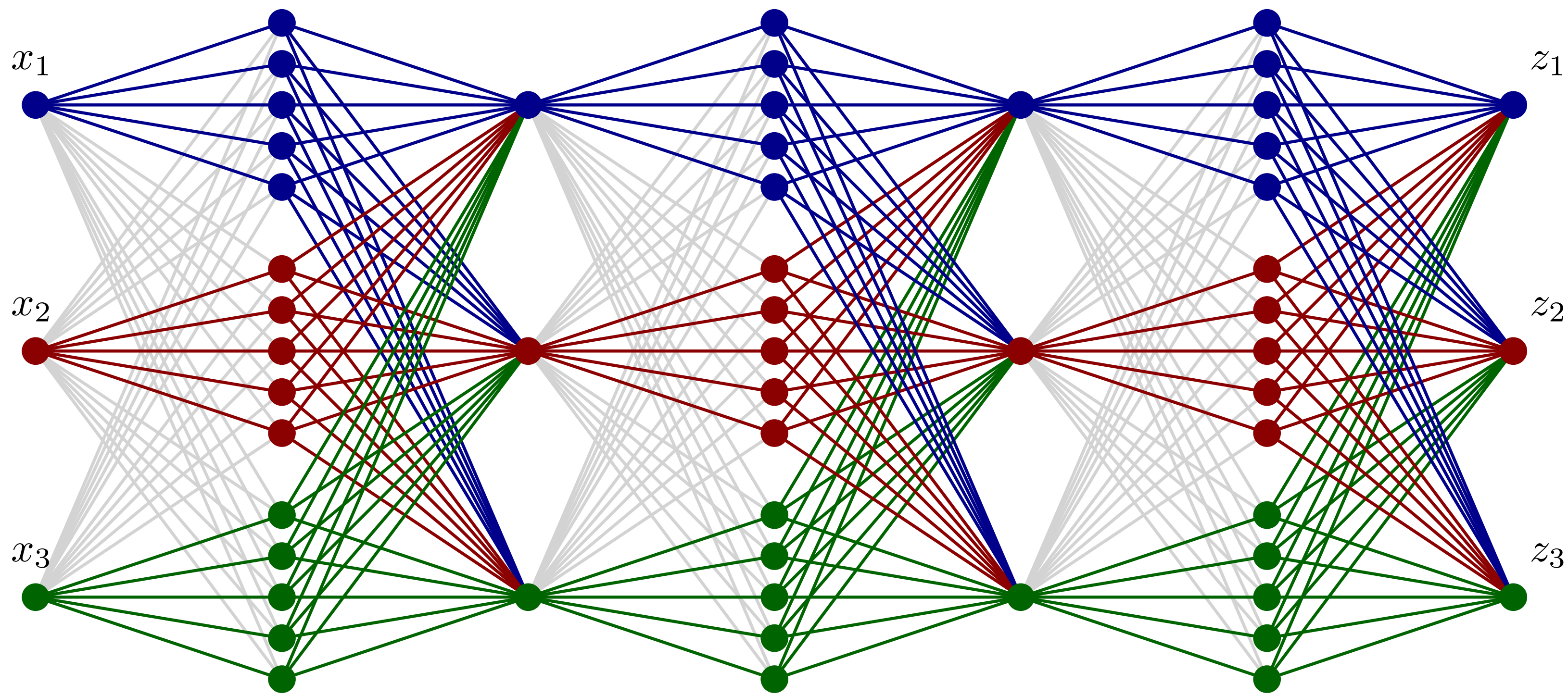
The Strong Lottery Ticket Hypothesis and the Random Subset Sum Problem



Frederik Mallmann-Trenn

King's College London

30 June 2025

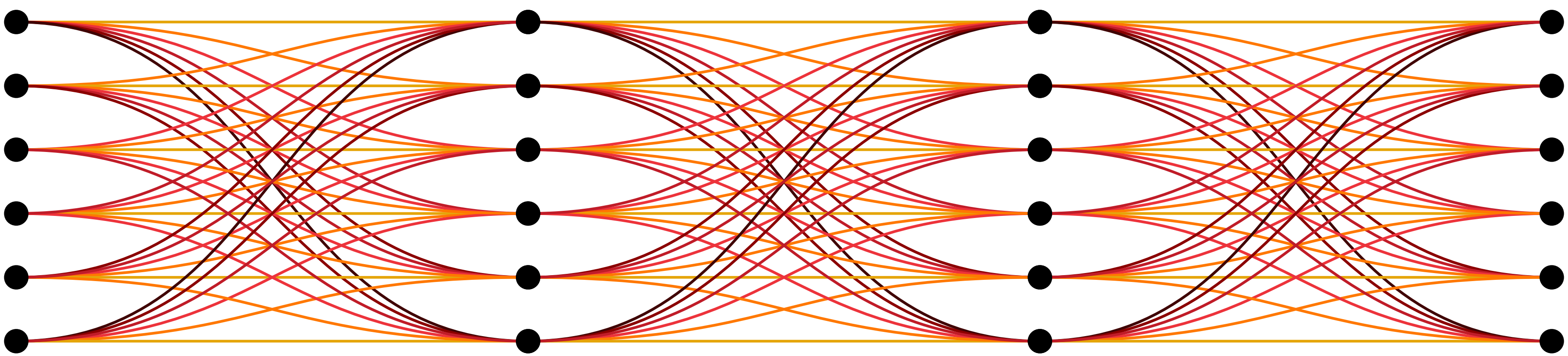
The slides are based on Francesco d'Amore's (Gran Sasso Science Institute in Italy) slides

Thank you so much!

Artificial neural networks are large

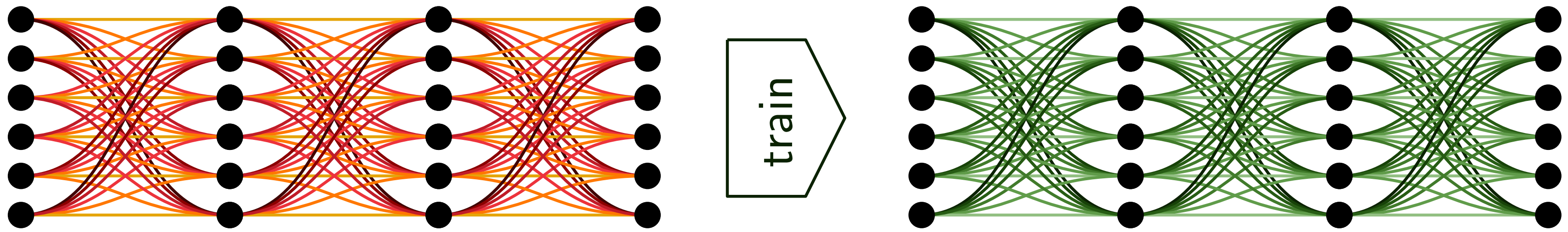
Usually ranging from millions to hundreds of billions parameters

- RESNET-50: > 20 millions parameters [He et al. 2015]
- BERT: > 100 millions parameters [Devlin et al. 2018]
- GPT-3: > 100 billions parameters [Brown et al. 2020]



Training and Inference are expensive

- Training large and dense networks yields good results
- However, it is very resource intensive



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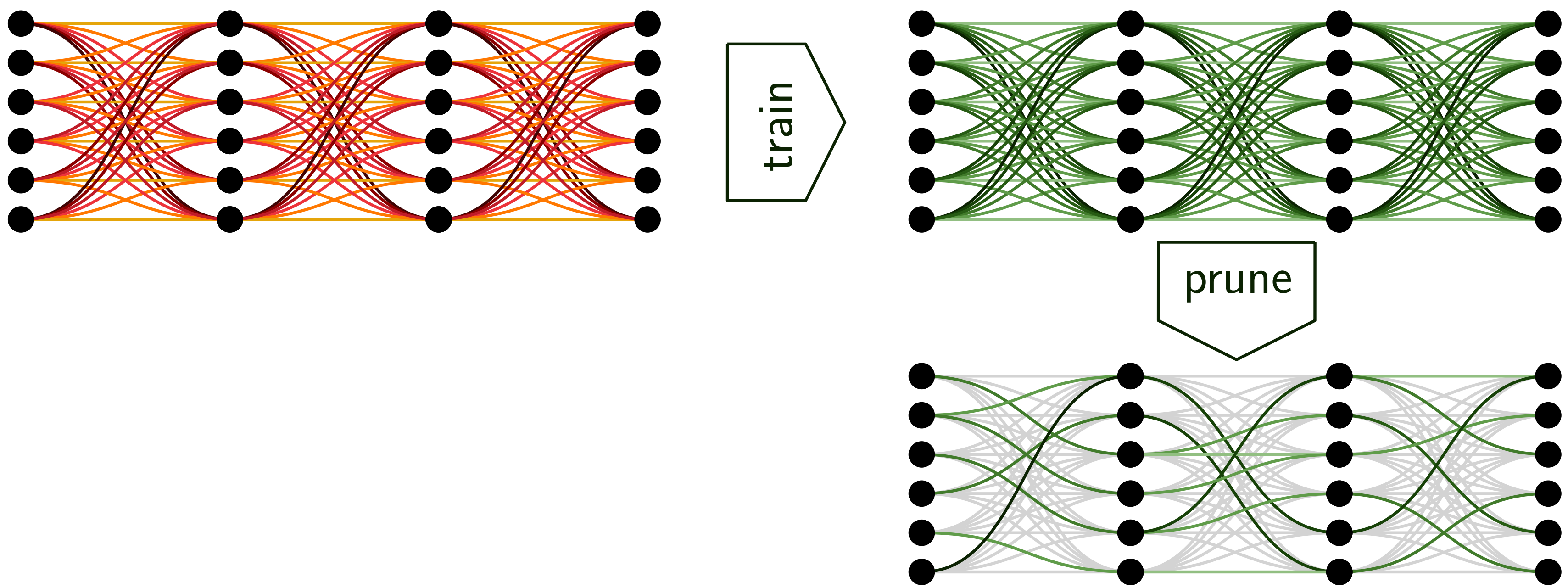

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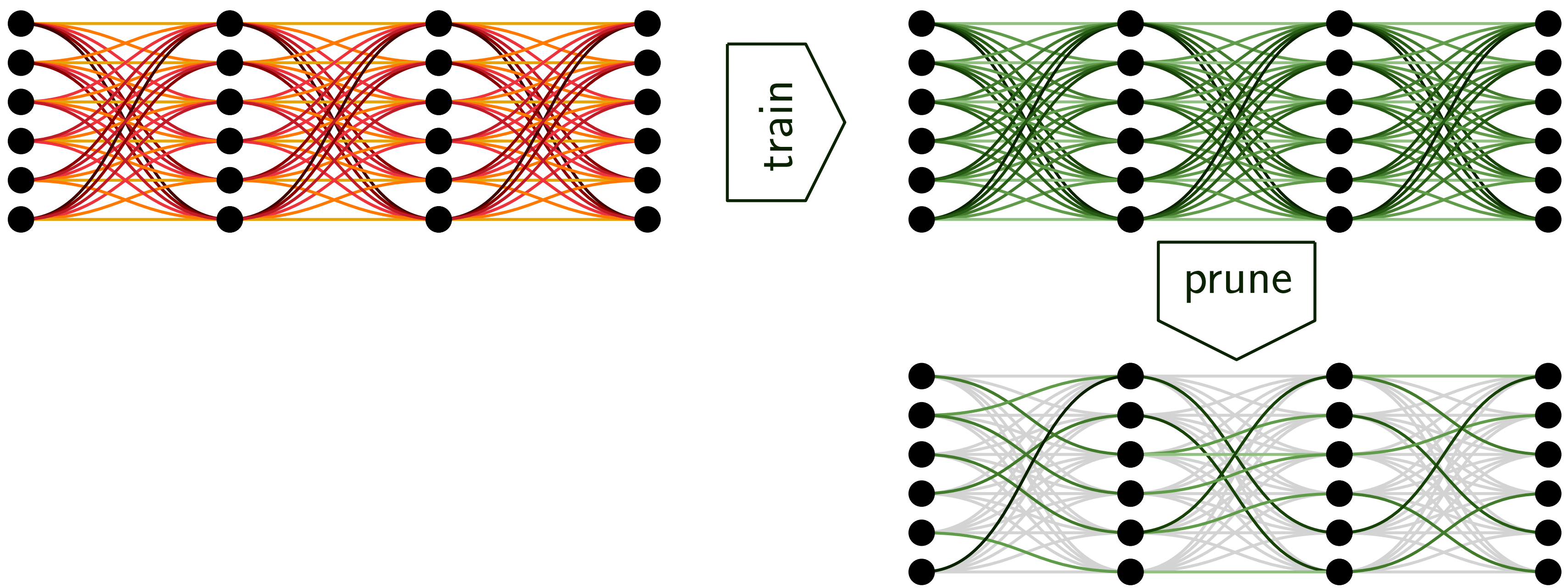

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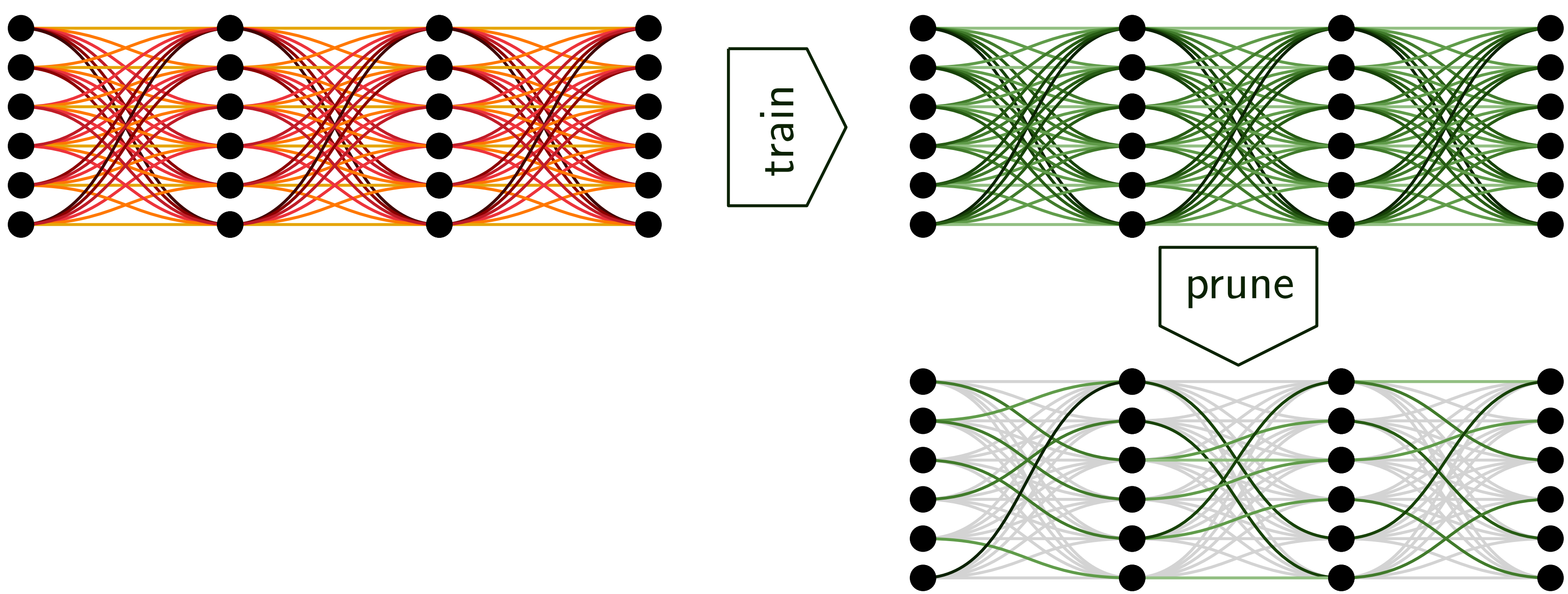
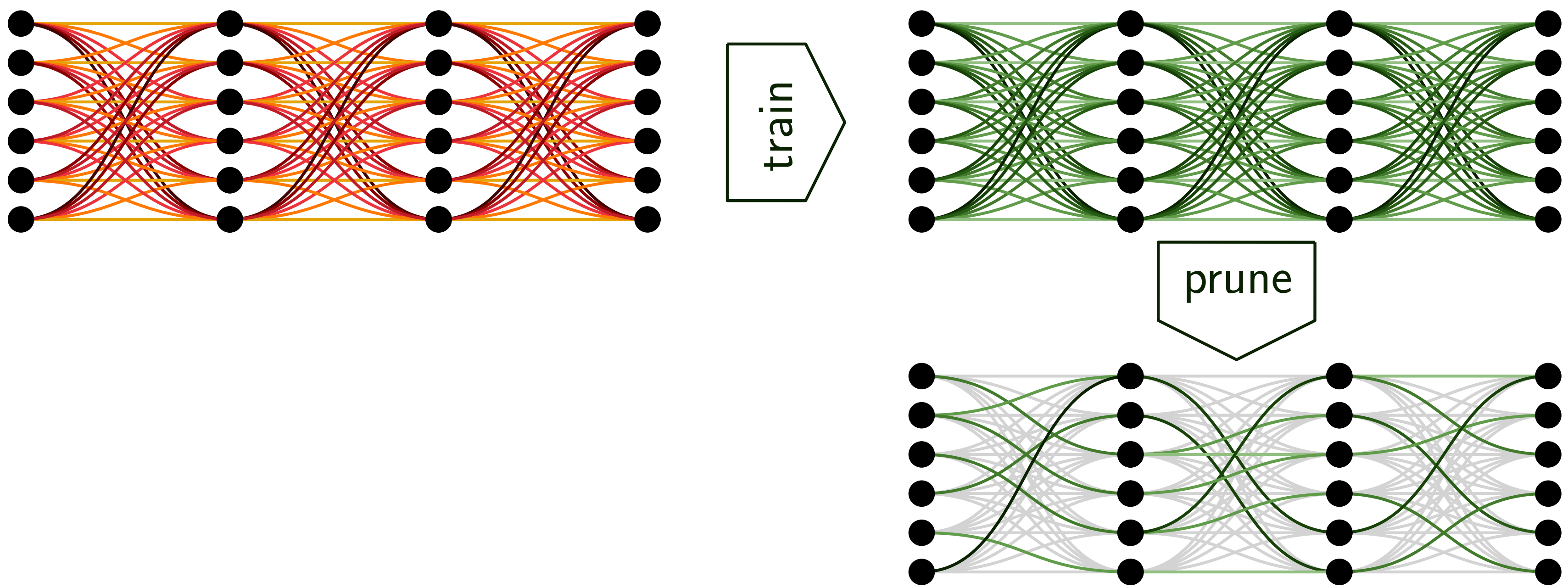
- Pruning $\sim 99\%$ of the edges can perform well [Hoefler et al. 2021]



What if we train the tiny one?

- Maybe, we can avoid the effort of dense training



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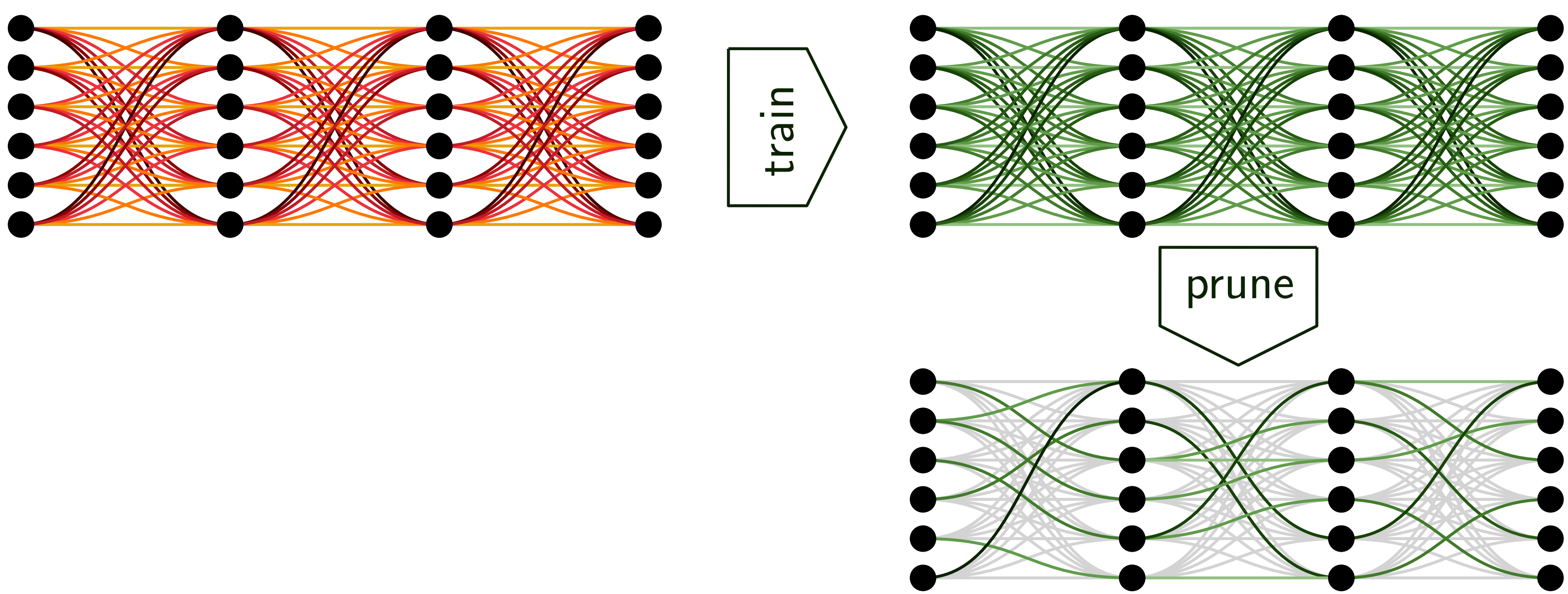
- **The Lottery Ticket Hypothesis:**
  "A randomly-initialized, dense neural network contains a subnetwork that is initialized such that—when trained in isolation—it can match the test accuracy of the original network after training for at most the same number of iterations."

[Frankle and Carbin 2019, ICLR]



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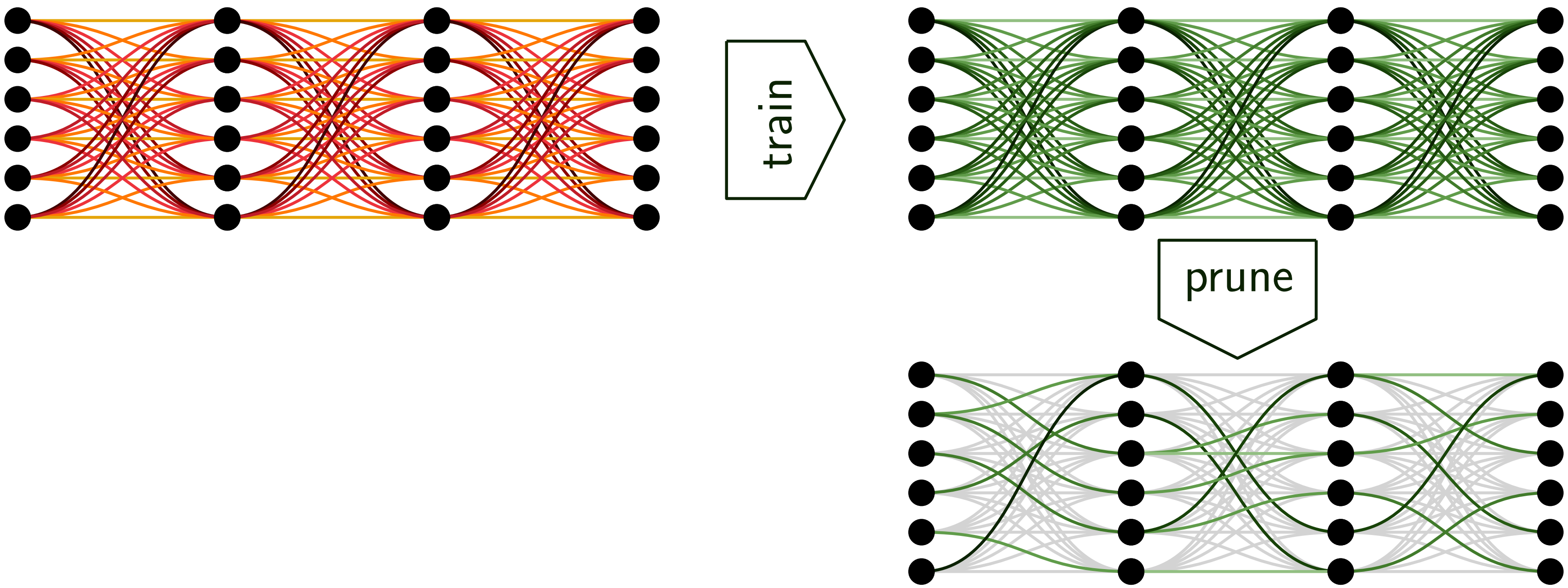
- To validate this hypothesis, let's train a dense network, then prune it



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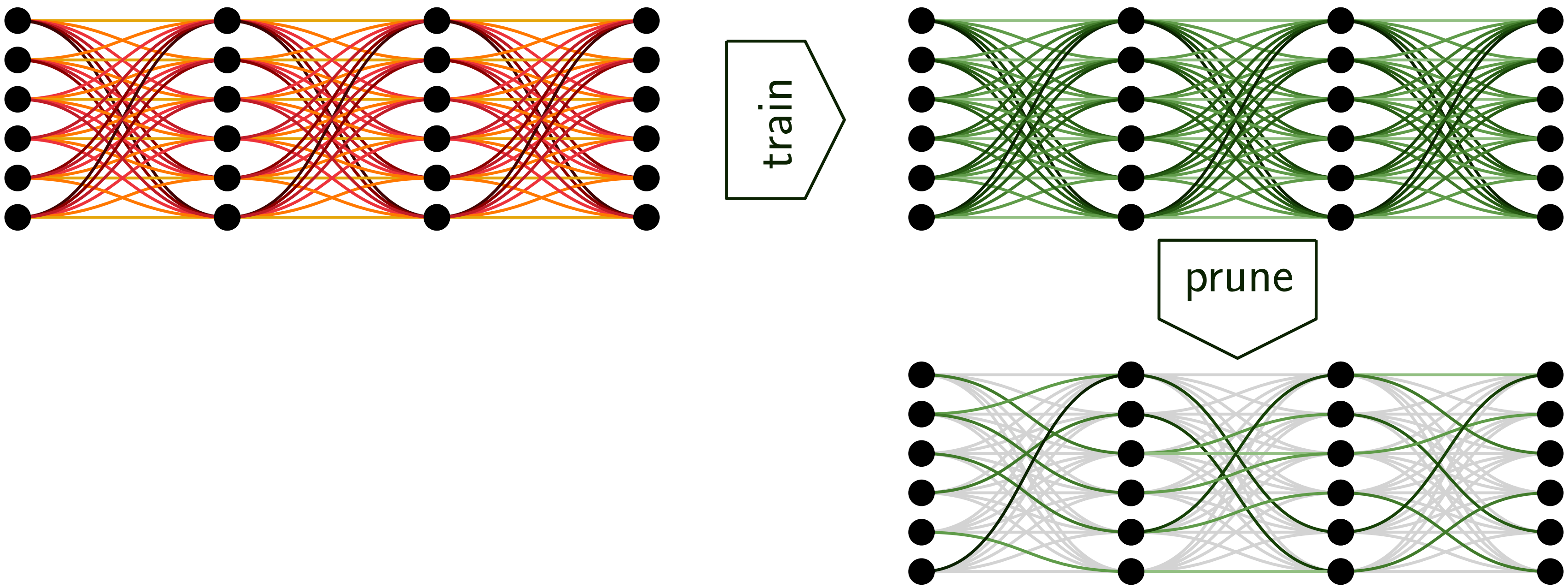

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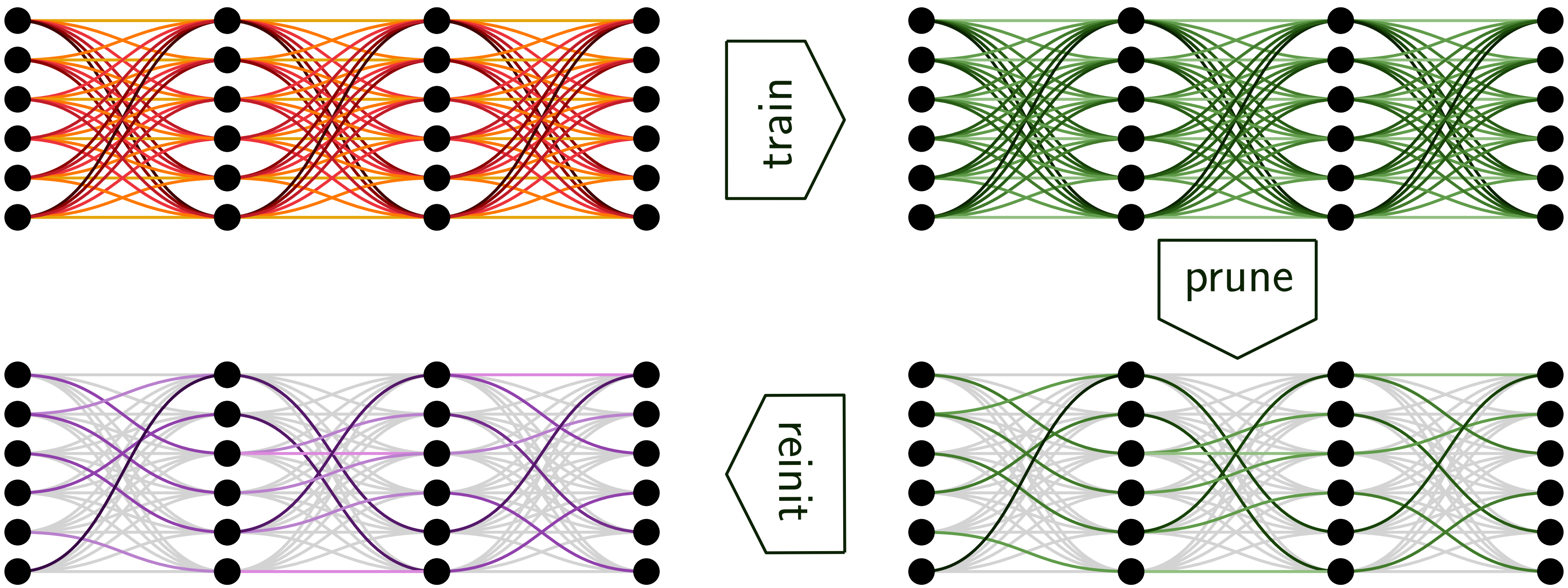

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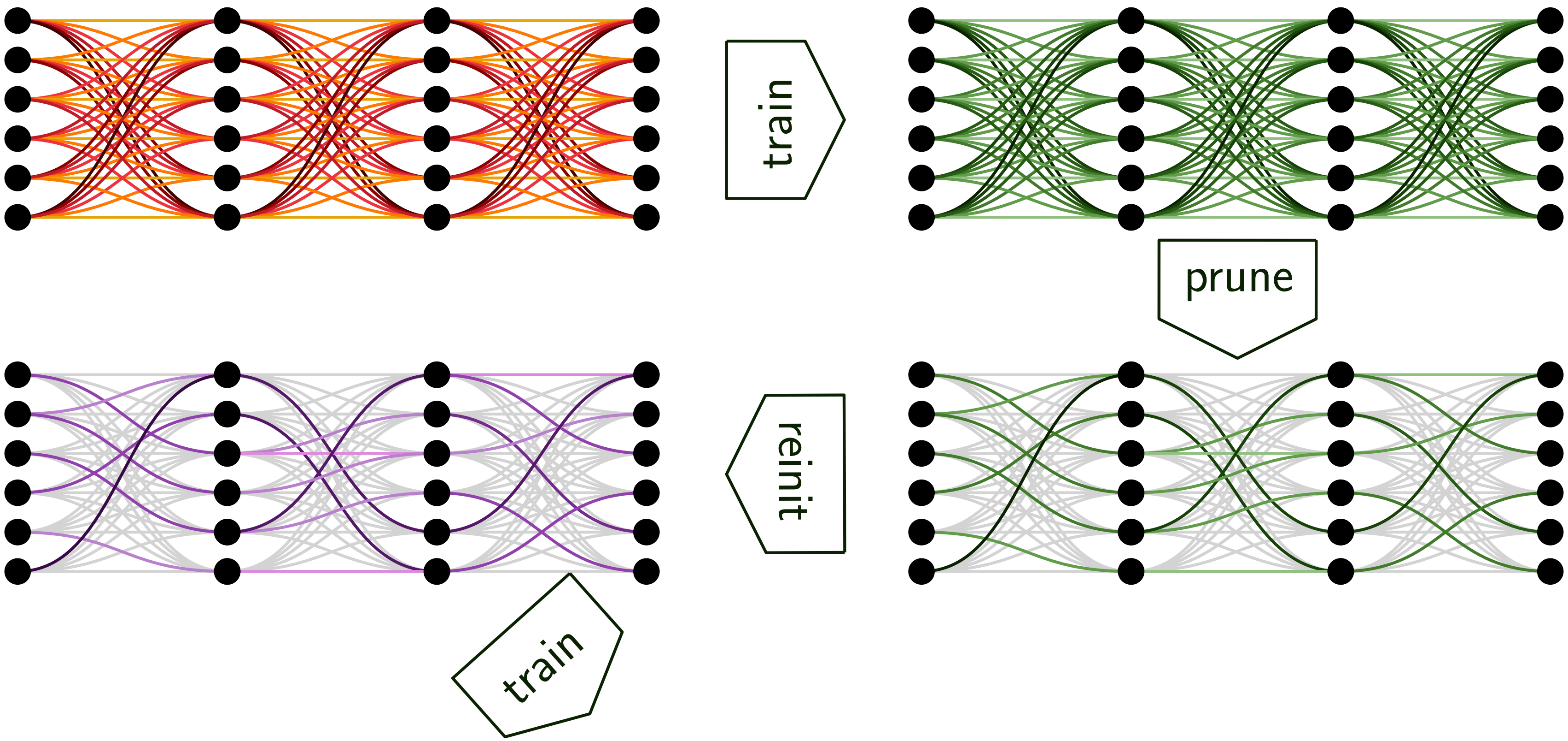

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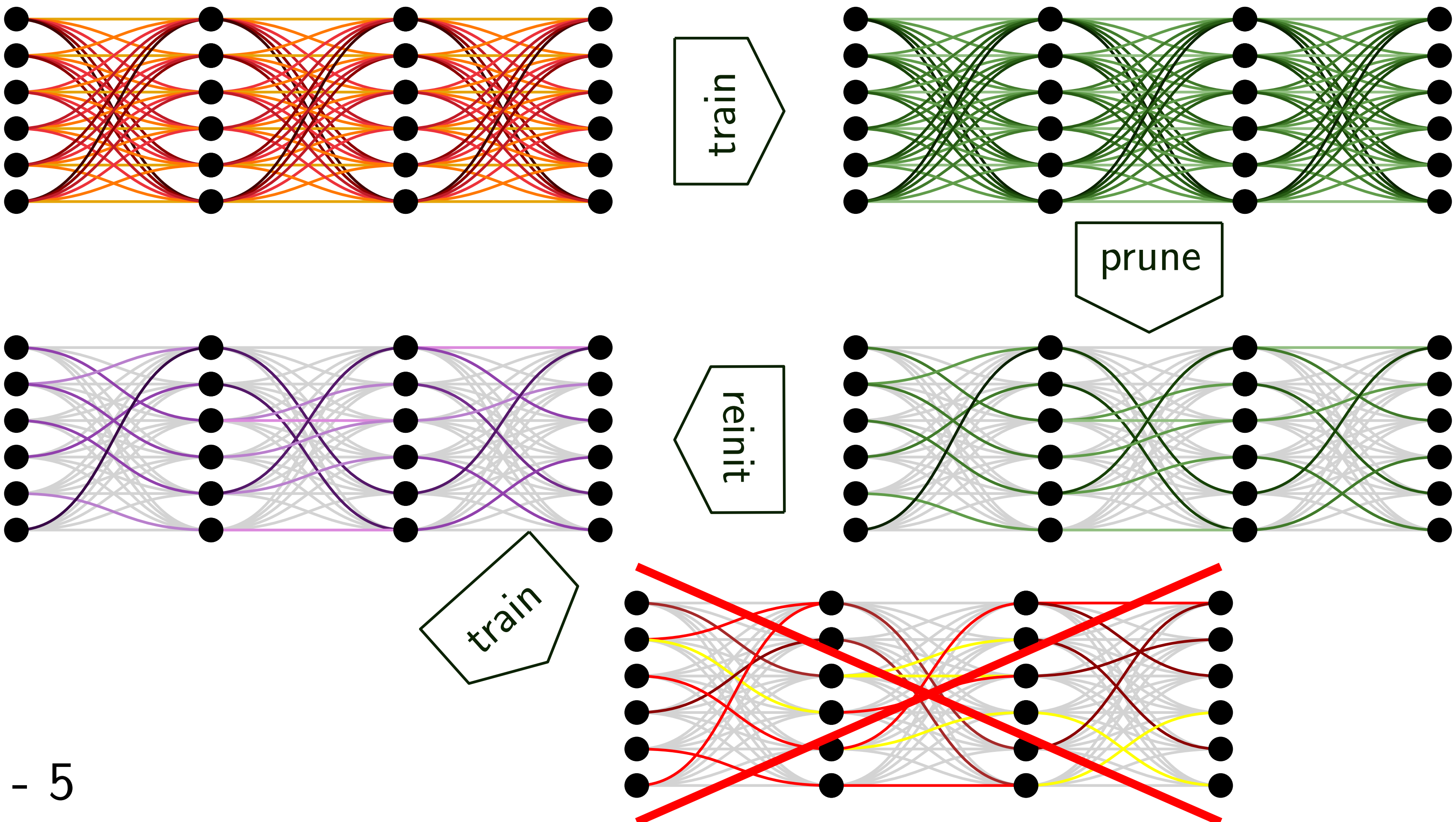
    - Bad accuracies

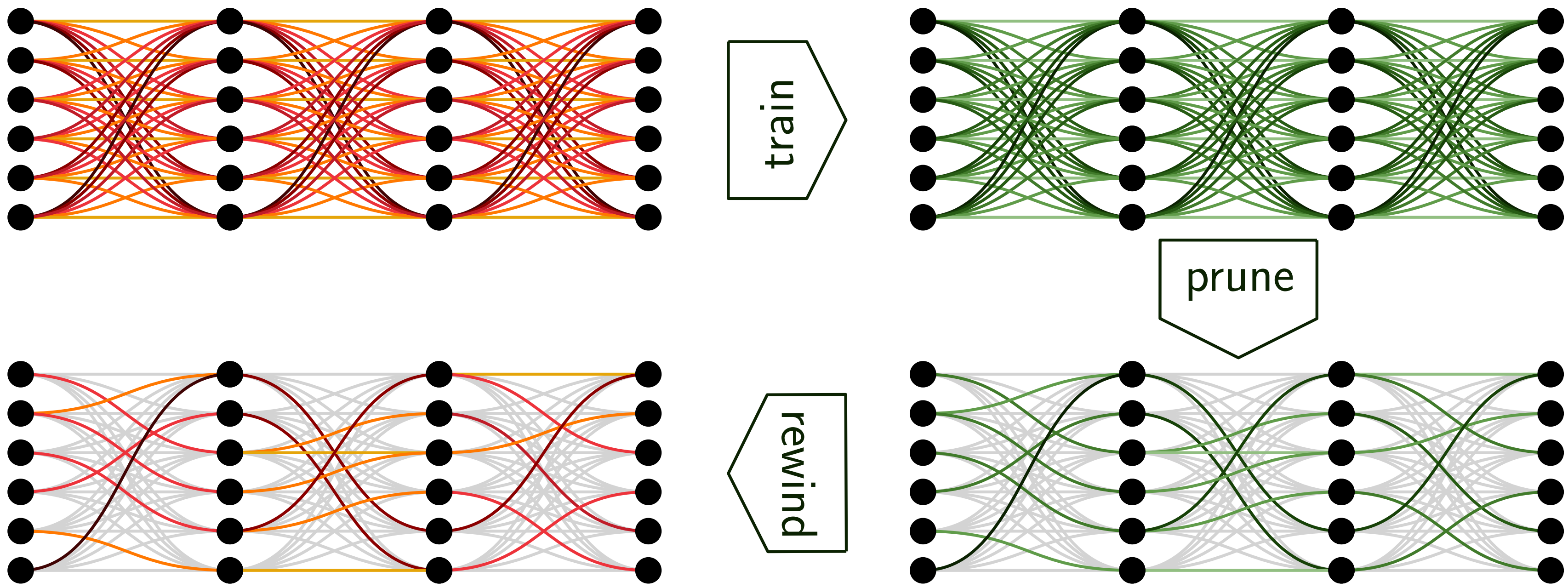


Not reinitialization, rewind instead

- Starting from a random point might be too much

[Frankle and Carbin 2019, ICLR]

- Rewind instead (back to initial random weights)



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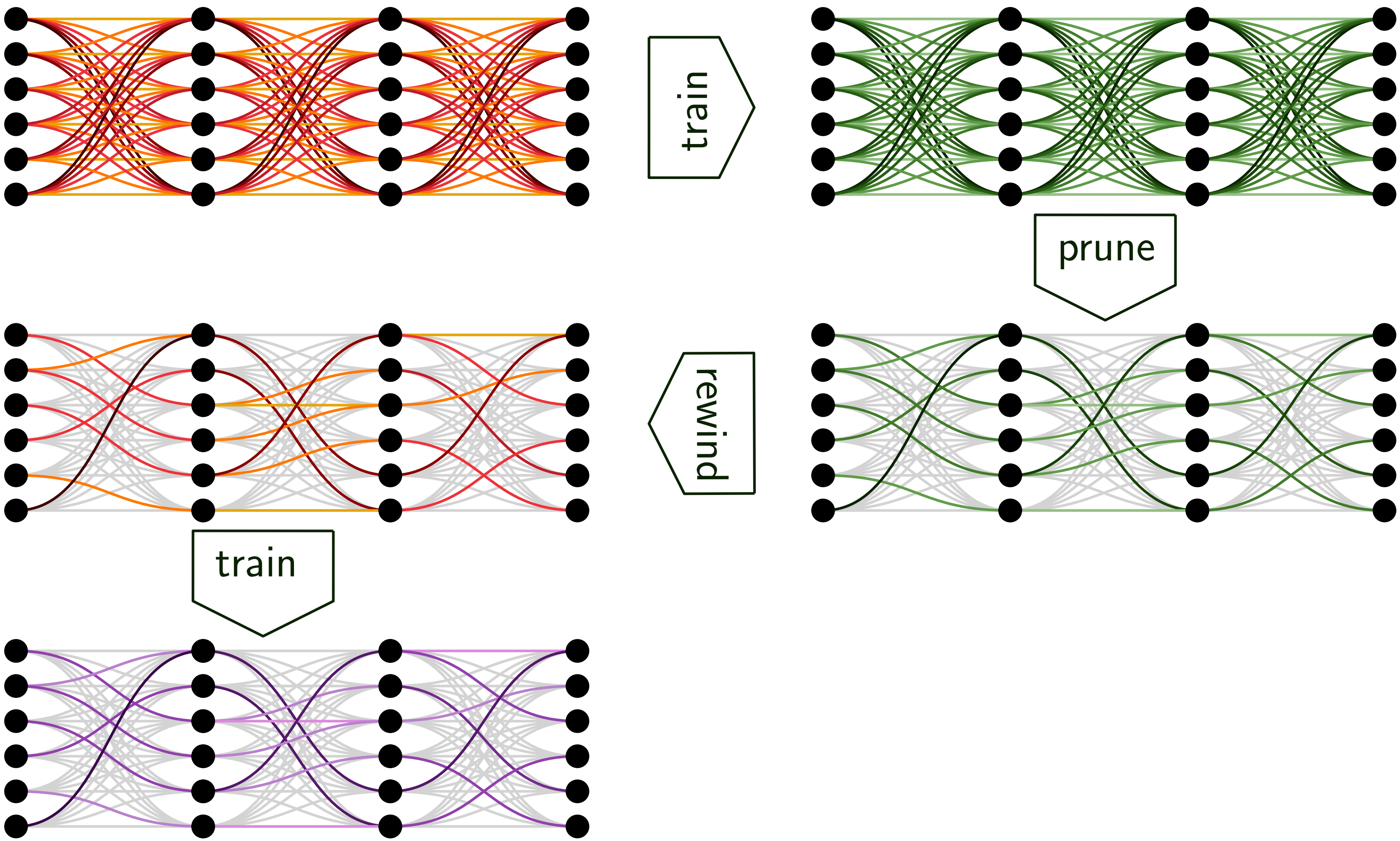

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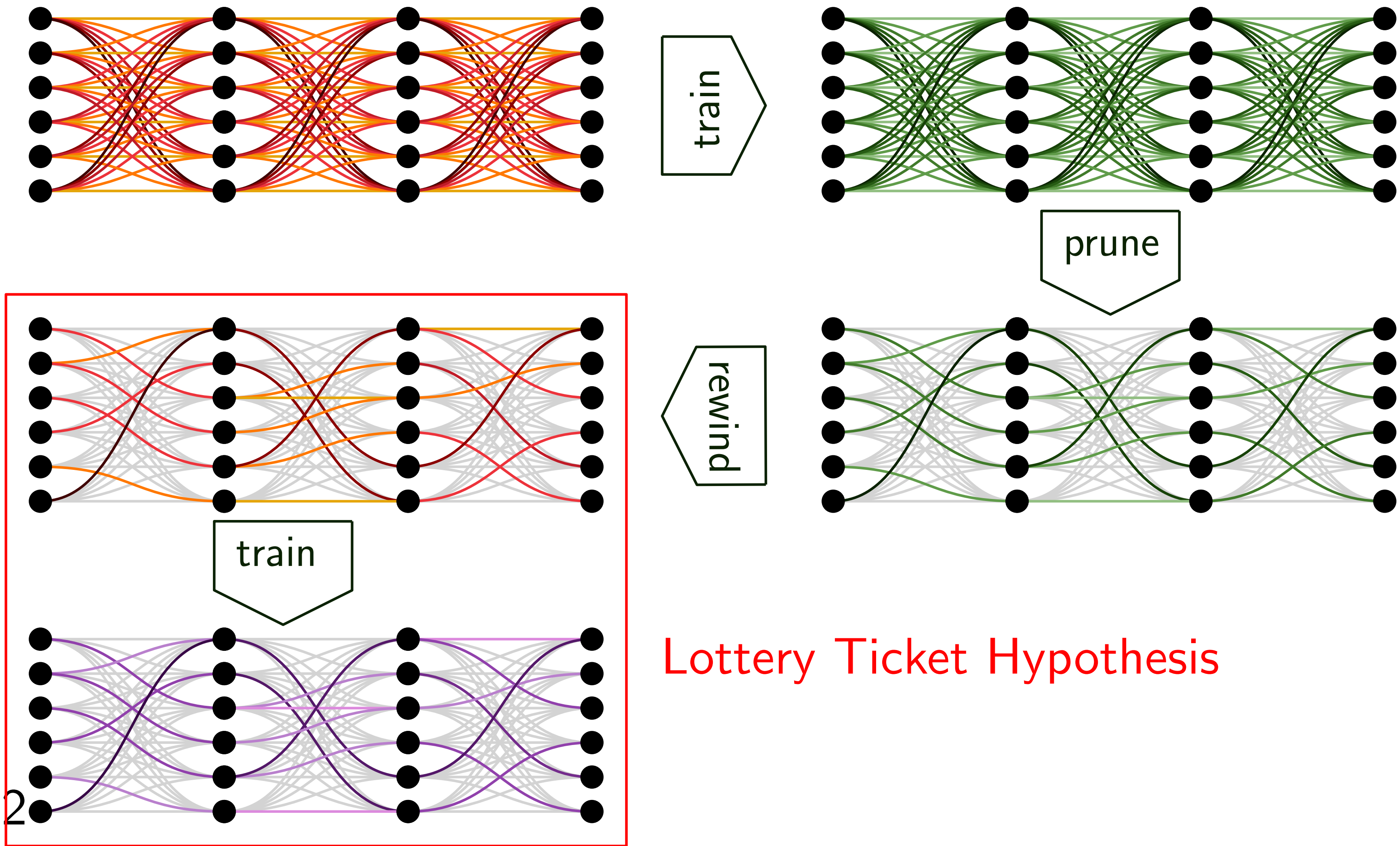

Lottery Ticket Hypothesis

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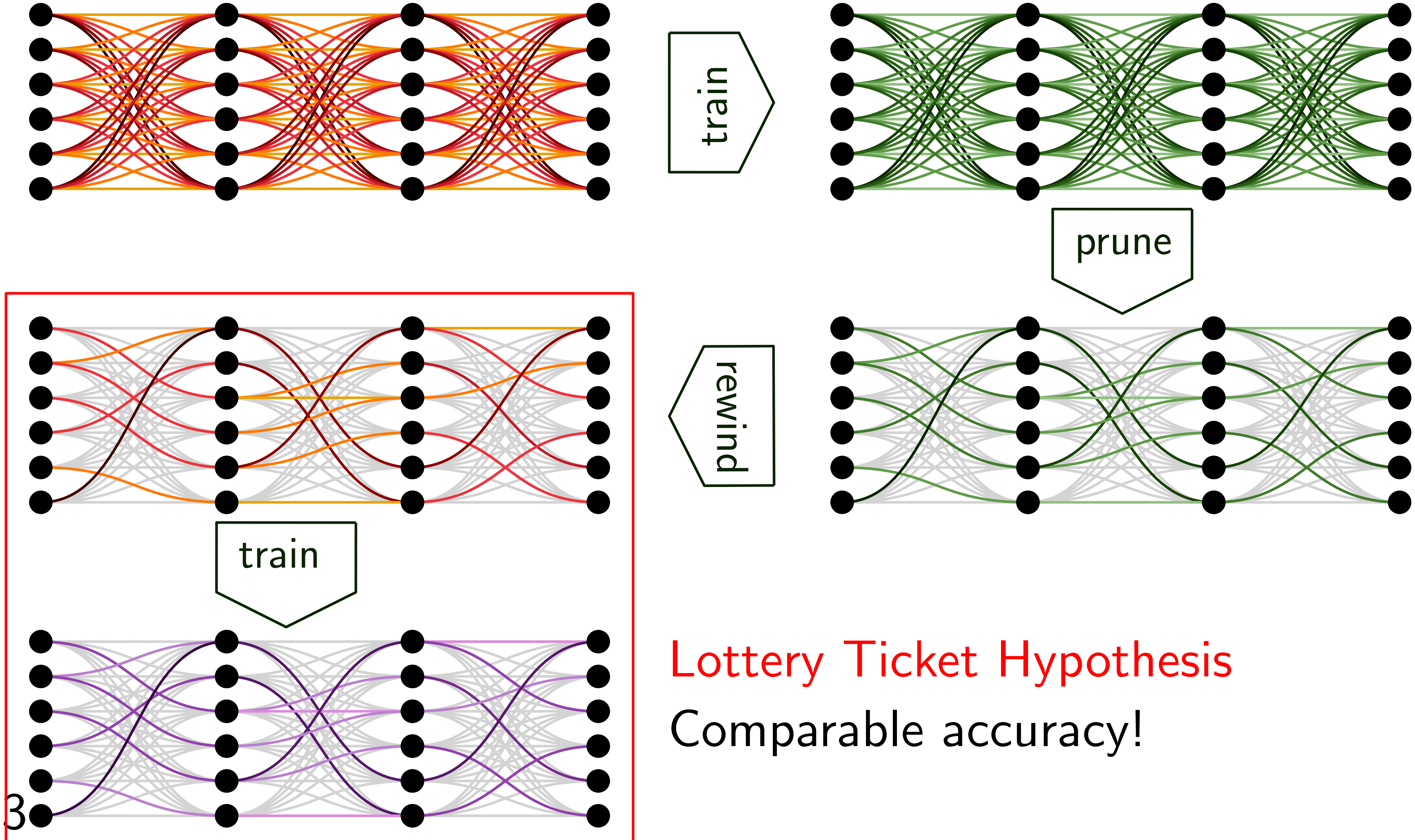

Lottery Ticket Hypothesis

Comparable accuracy!

Lottery tickets

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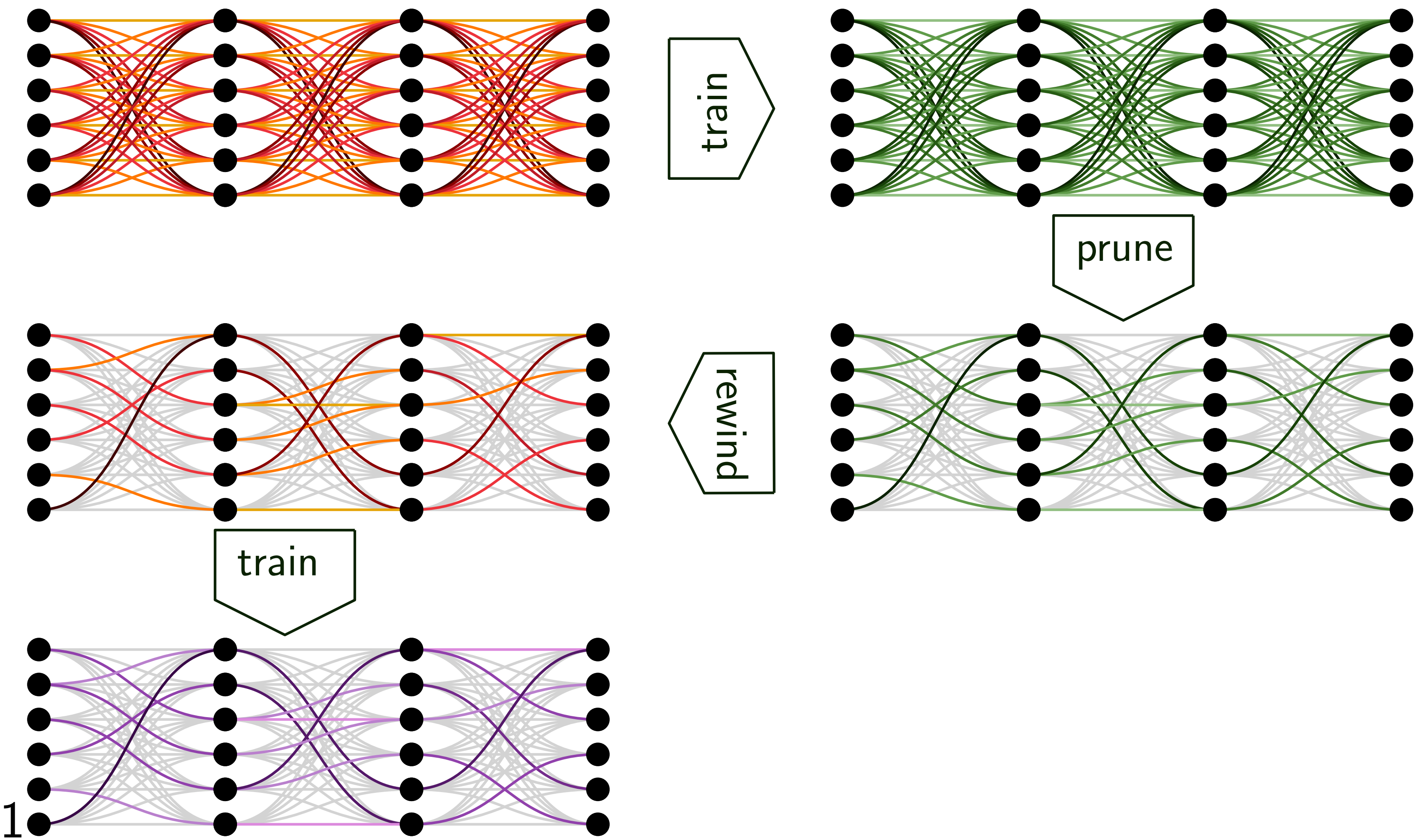

Lottery tickets

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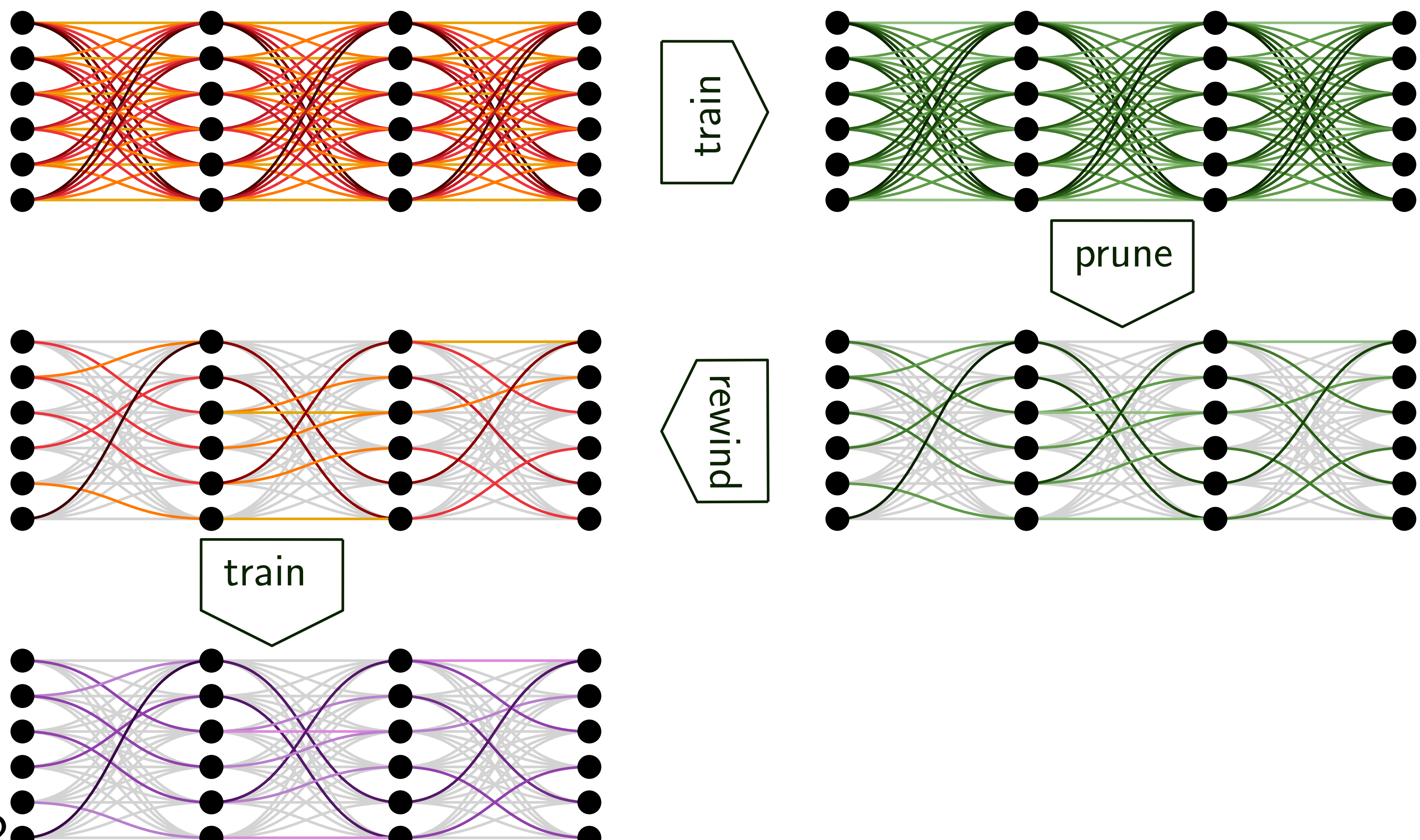

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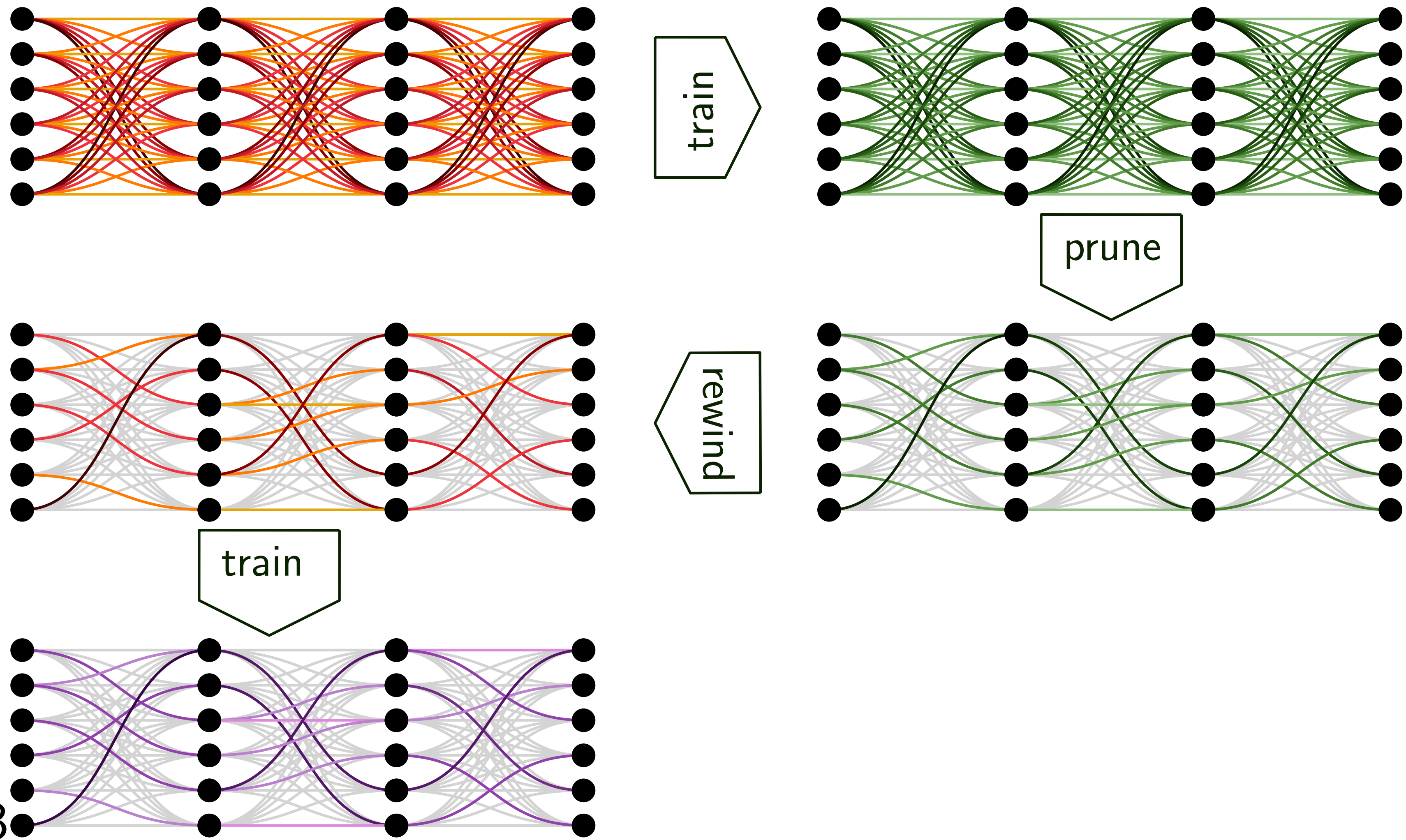

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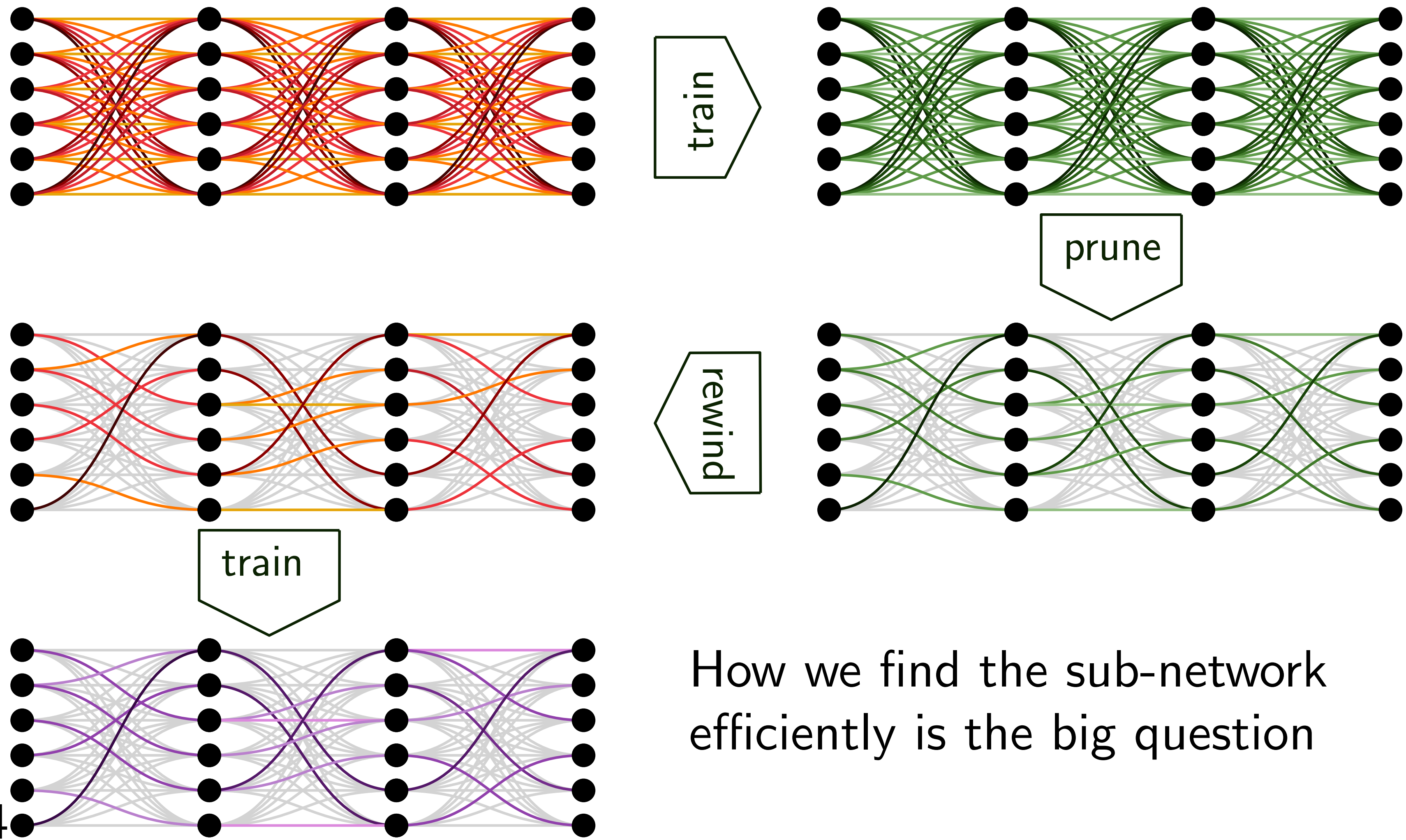
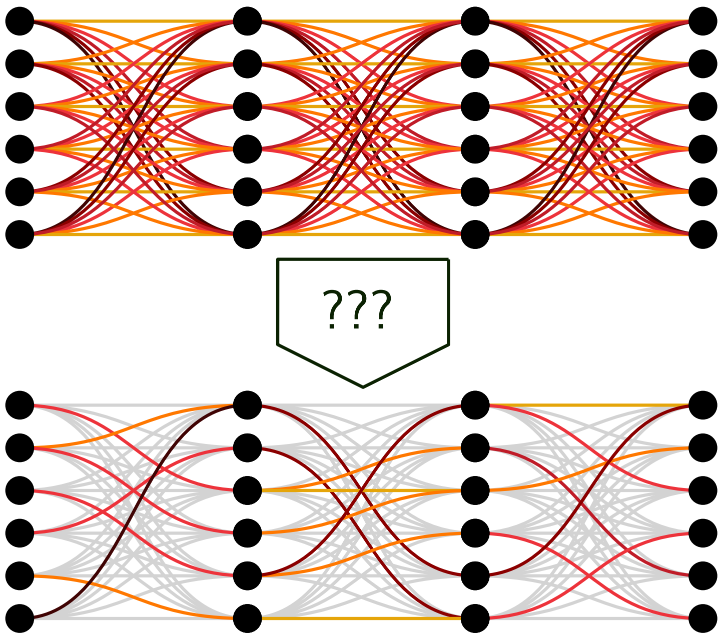


How we find the sub-network efficiently is the big question

The Lottery Ticket Hypothesis (LTH)

[Frankle and Carbin 2019, ICLR]: winning lottery tickets always exists



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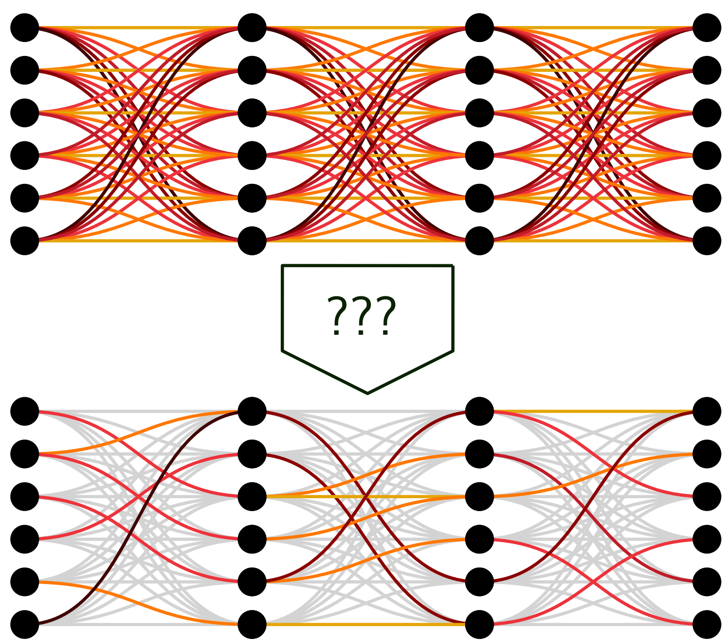

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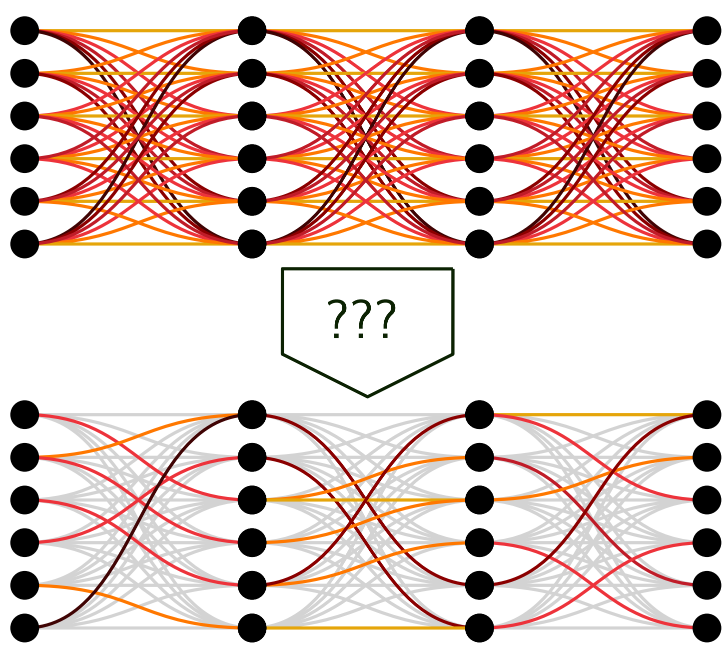

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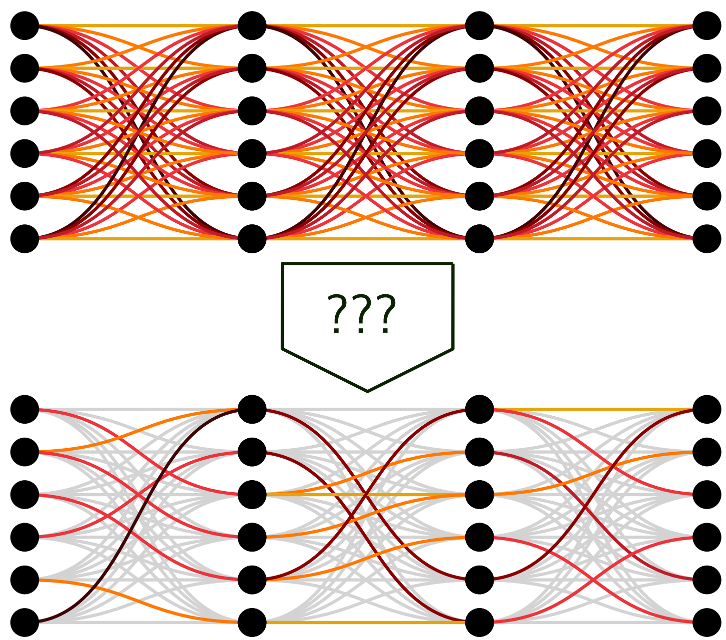
If we want to understand deep learning, we should probably understand this first.



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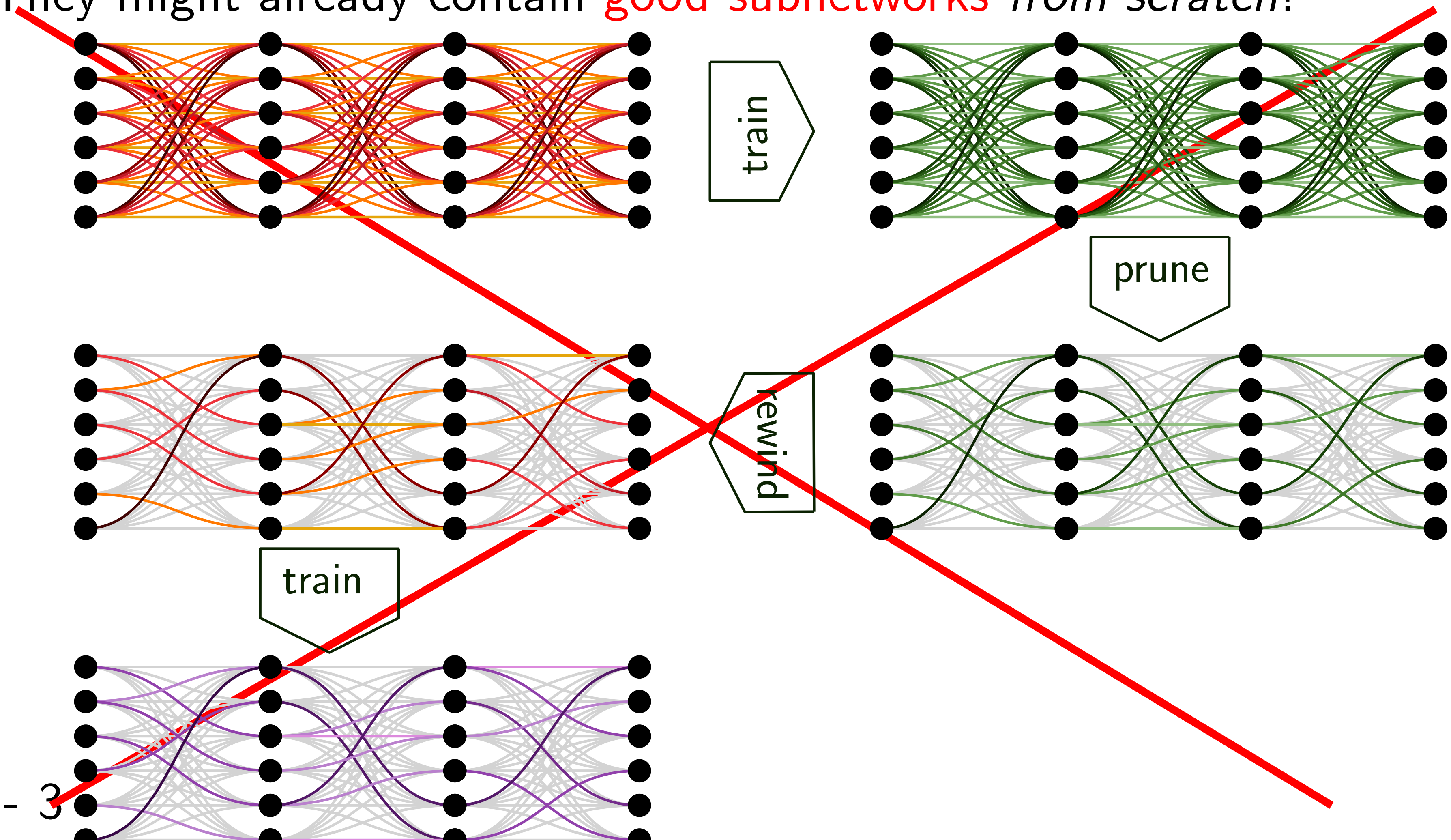

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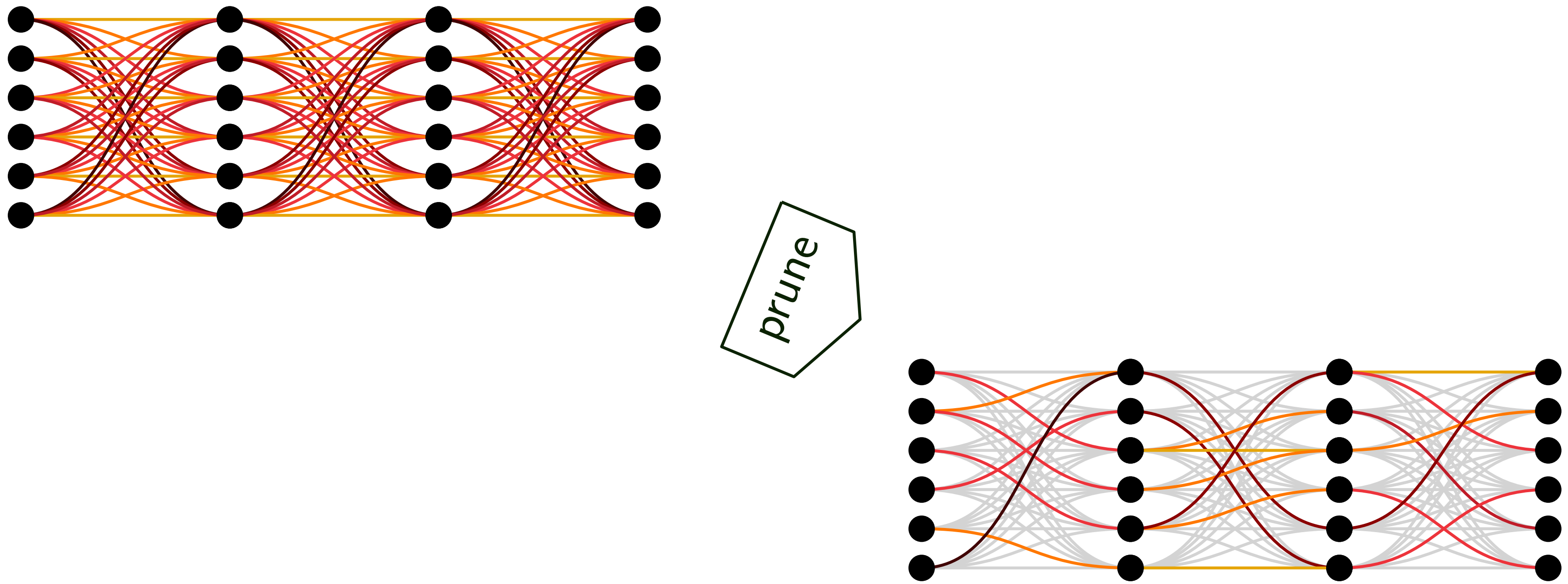

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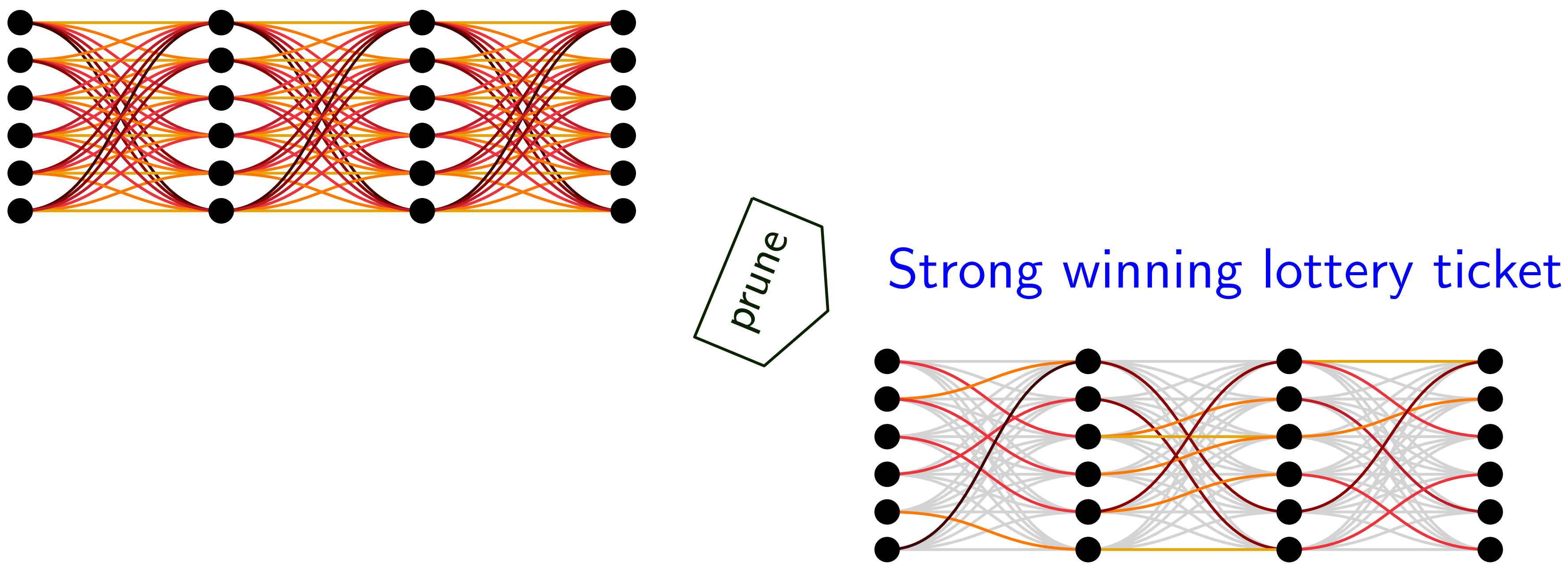

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[Diffenderfer and Kailkhura 2021, ICLR]: quantized strong winning lottery tickets in ResNet-50 (binary weights) outperform the original on ImageNet

Do we have a theorem?

Target result: *Let $\mathcal{F}$ be the class of neural networks with a given size. If a network g with random weights is sufficiently large, then, with high probability, it is possible to prune g to approximate any network in $\mathcal{F}$*

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- With high probability: $1 - \delta$ for any given $\delta > 0$
- Approximation: distance w.r.t. some metric is ε for any given $\varepsilon > 0$

Overview of the (theoretical) results

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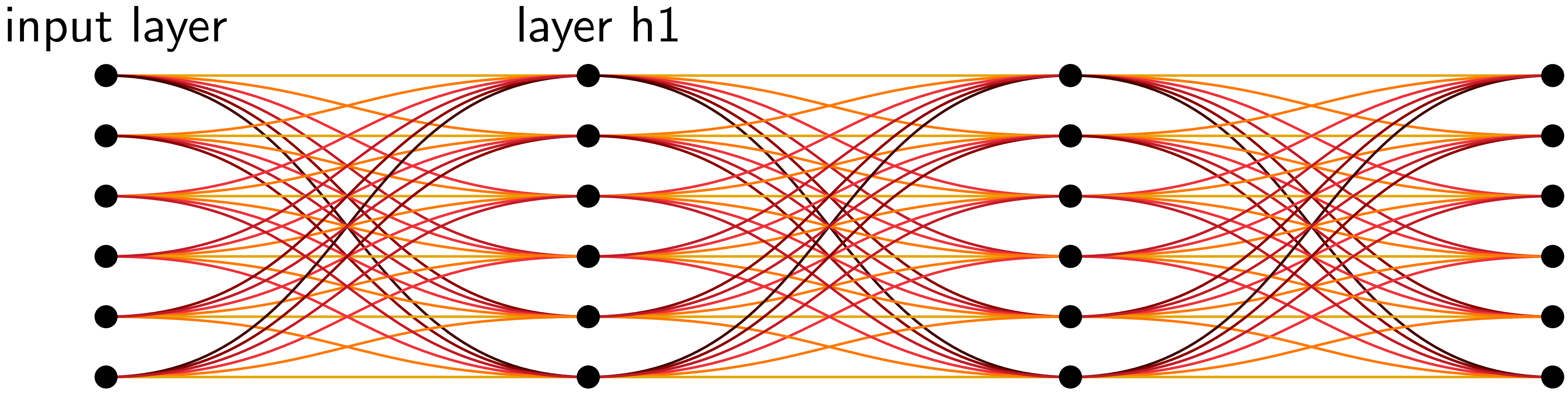
Review: Feedforward

Dense network: $f(\mathbf{x}) = \mathbf{W}_\ell \sigma(\mathbf{W}_{\ell-1} \ldots \sigma(\mathbf{W}_1 \mathbf{x}))$

- $\mathbf{x} \in \mathbb{R}^{d_0}$, $\mathbf{W}_i \in \mathbb{R}^{d_{i-1} \times d_i}$
- $\sigma(x) = \max(0, x)$ (ReLU)



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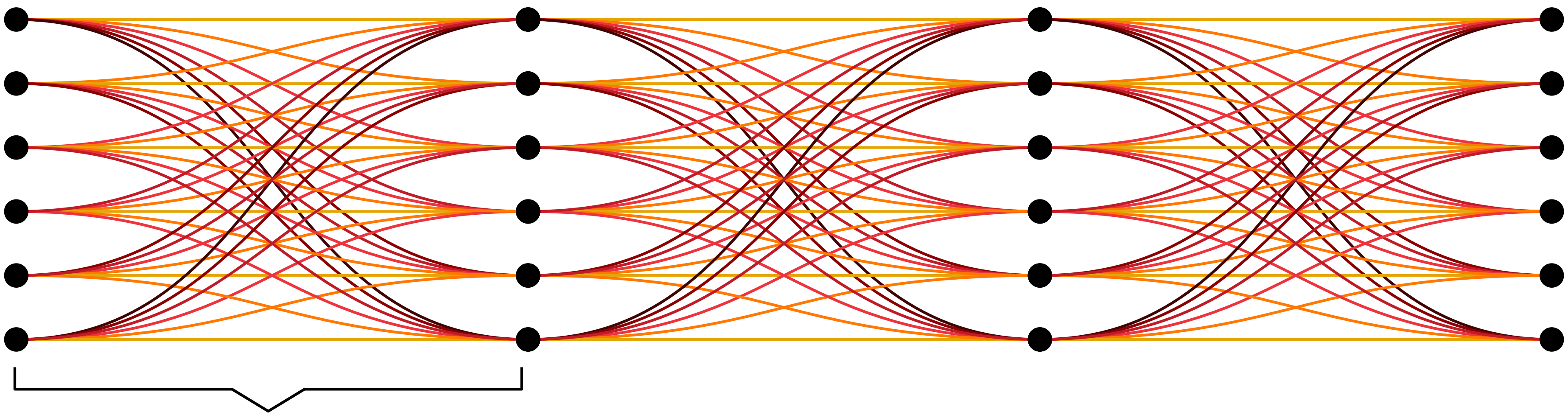

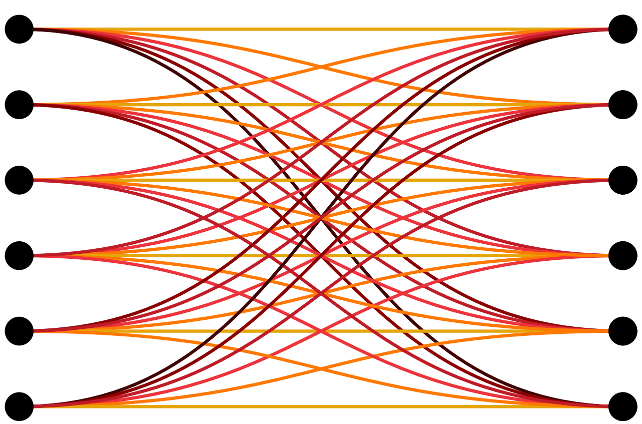

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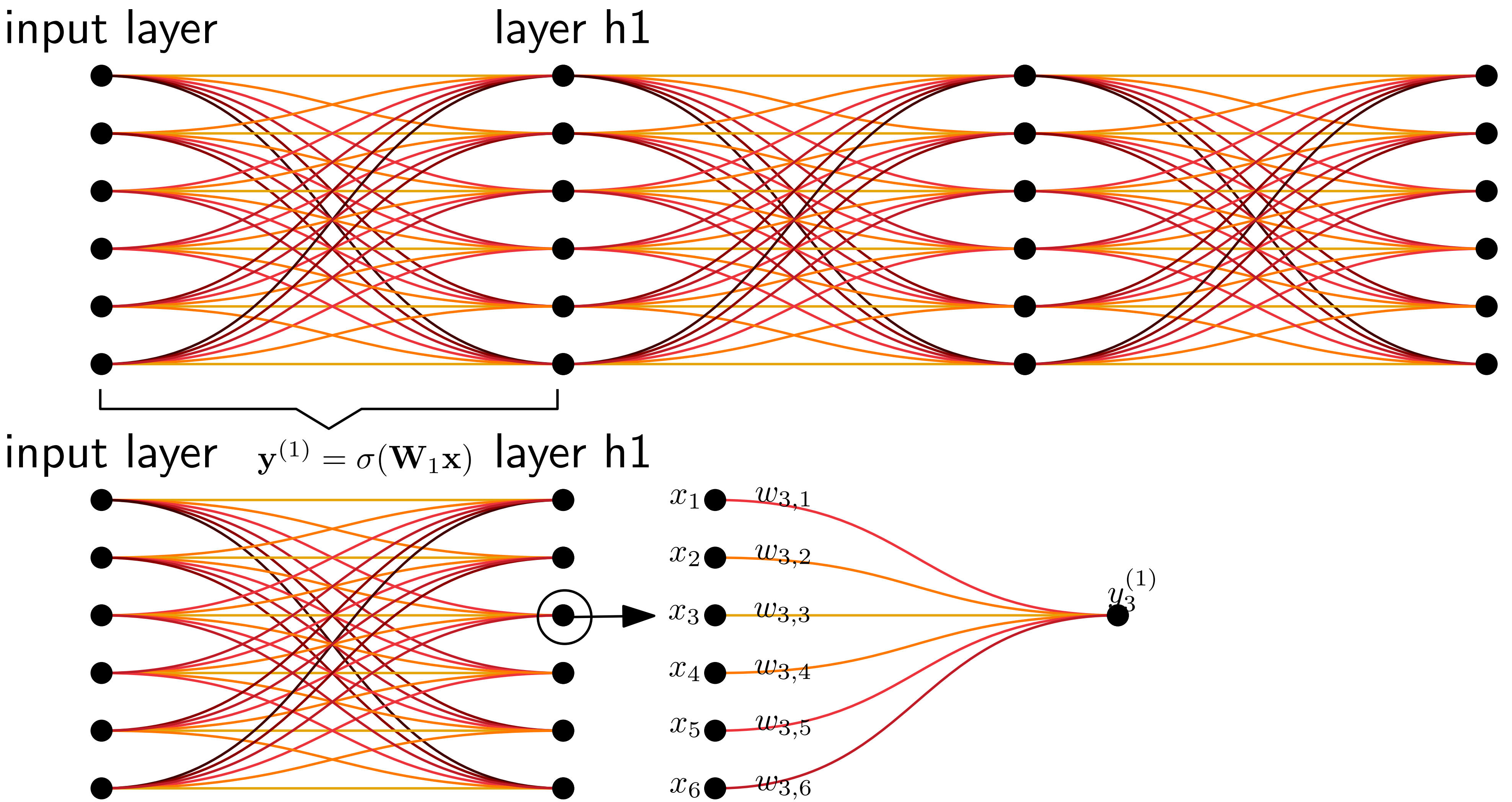

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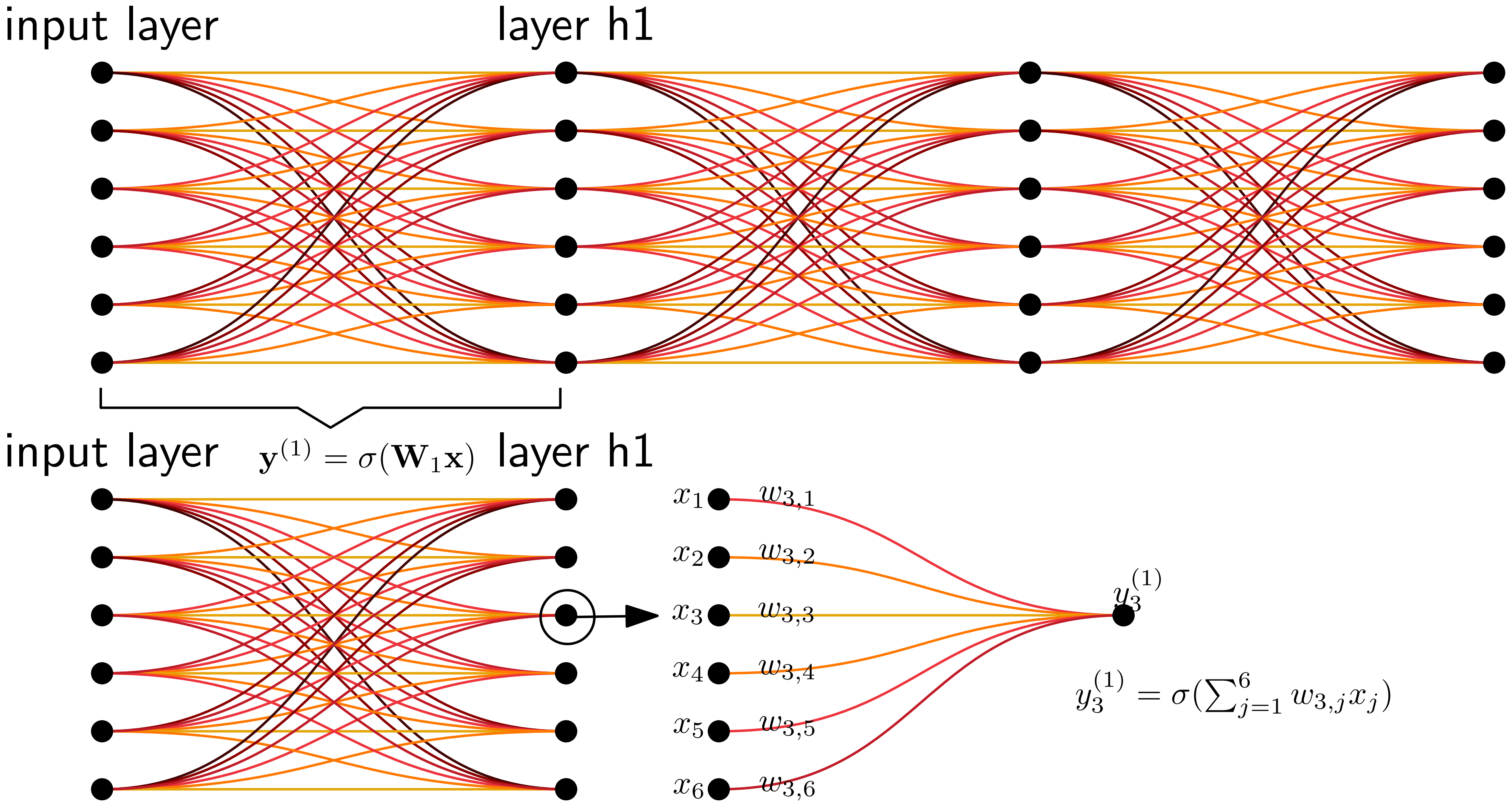
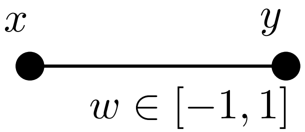


Review: SLTH in dense networks

- Approx one edge: approx $y = wx$ for all x within error ε (no ReLU)



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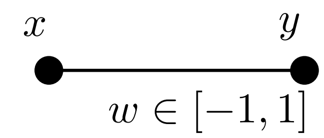

  - Naïve approach

    sample many weights $w_i \sim$ Unif$[-1, 1]$
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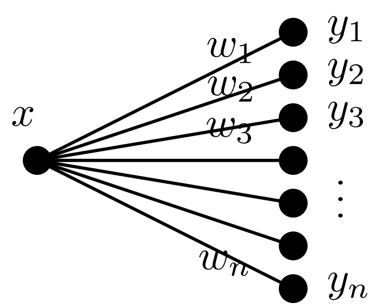

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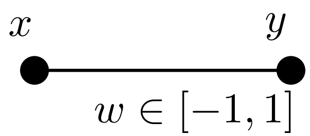

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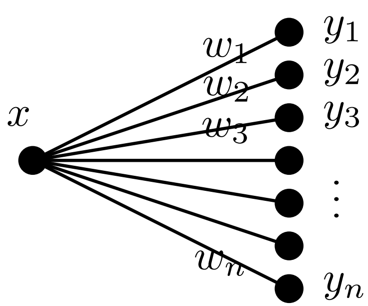

    roughly $1/\varepsilon$ samples

Review: SLTH in dense networks

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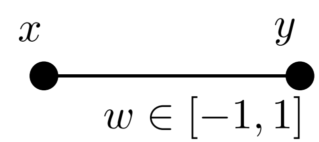
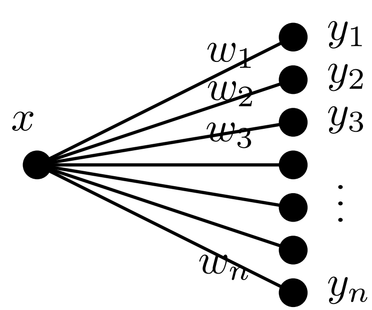


- Naïve approach

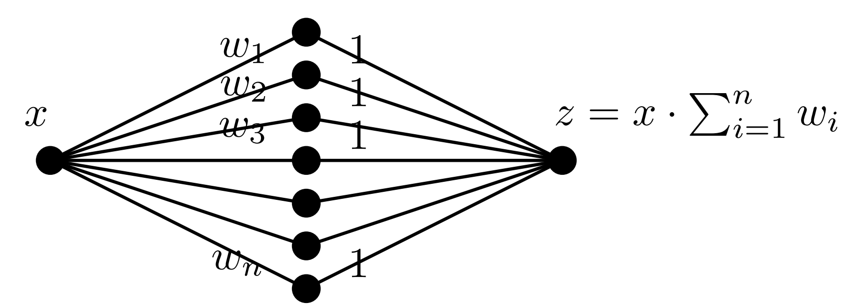
  sample many weights $w_i \sim$ Unif$[-1, 1]$
  until getting w, and prune the others



  roughly $1/\varepsilon$ samples

- Random subset sum (RSS) approach:

  add intermediate layer, sample
  $w_i \sim$ Unif$[-1, 1]$ and find a good subset



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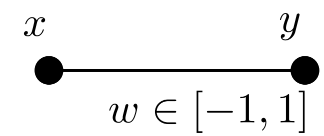

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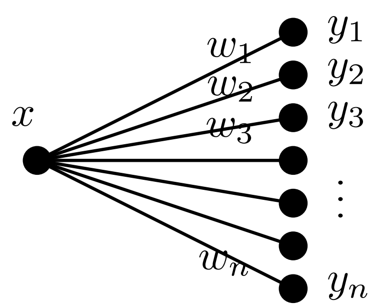

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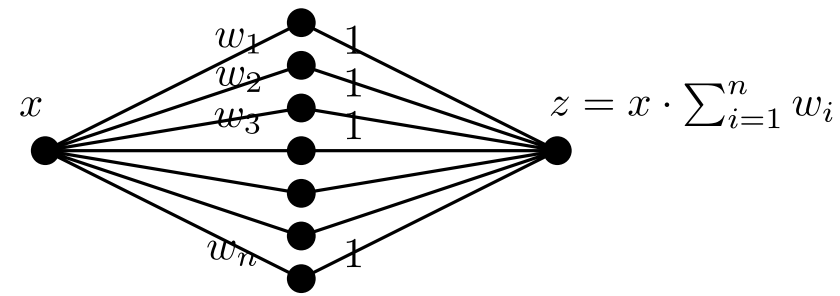

Say $w = 0.5$, $w_1 = 0.6$, $w_2 = -0.1$, ...

Review: SLTH in dense networks

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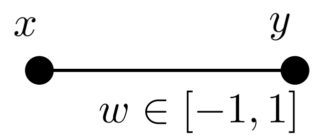

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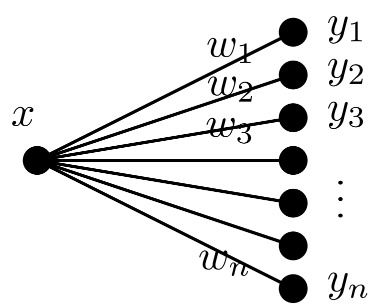

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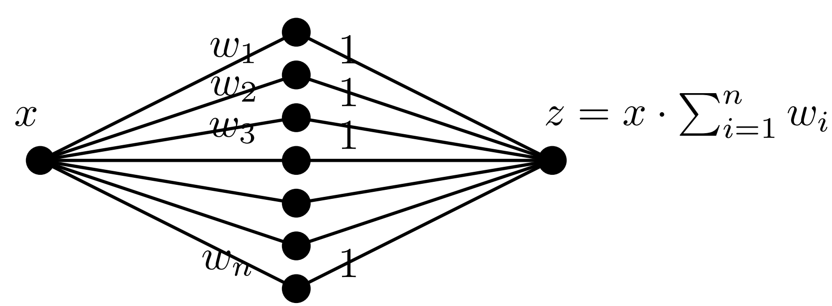

  How many?

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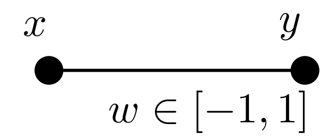

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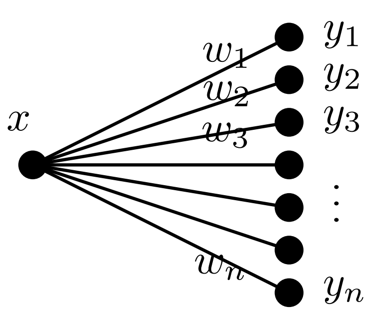

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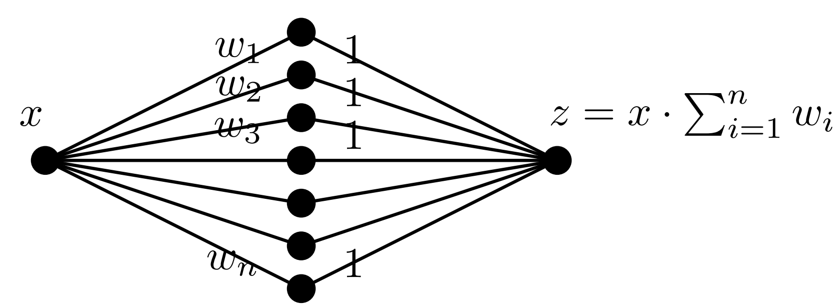

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Theorem [Lueker 1998; da Cunha et al. 2023, ESA]: *Let $x_1, \ldots, x_n \in [-1, 1]$ be i.i.d. uniform random variables. Given any error parameter $\varepsilon > 0$, there exists a constant $C > 0$ such that if $n \geq C \log 1/\varepsilon$ then, with probability $1 - \exp\left[(n - C \log 1/\varepsilon)^2 / 4n\right]$, for each $z \in [-1, 1]$ there exists a subset $S \subseteq [n]$ such that $\left|z - \sum_{i \in S} x_i\right| < 2\varepsilon$*

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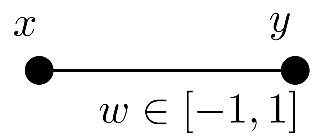

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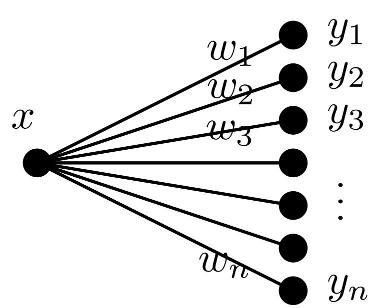

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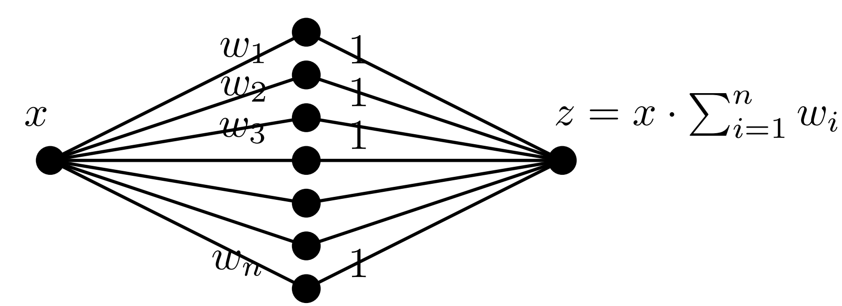

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works for all densities $h(x) = pf(x) + (1 - p)g(x)$, where f is "uniform"

Apply RSS for pruning

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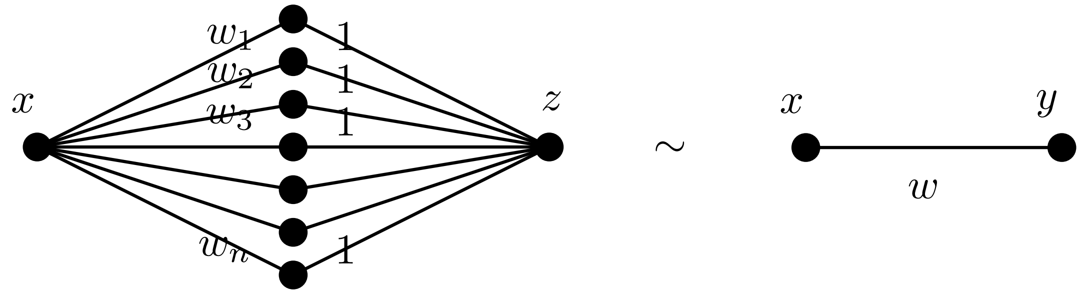

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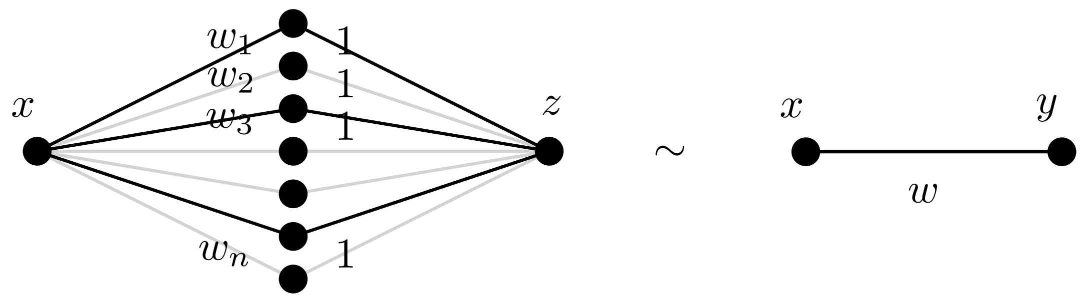

$n \geq C \log 1/\varepsilon \implies \exists S \subseteq [n] : \left|w - \sum_{i \in S} w_i\right| < 2\varepsilon$

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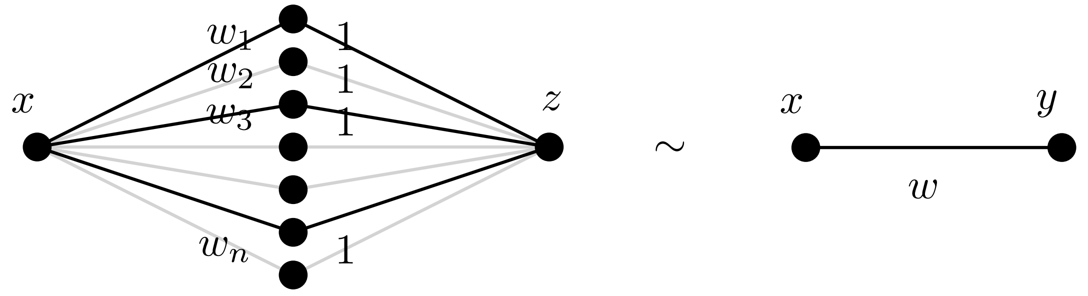

$n \geq C \log 1/\varepsilon \implies \exists S \subseteq [n] : \left| w - \sum_{i \in S} w_i \right| < 2\varepsilon$

$\implies \left| wx - \sum_{i \in S} w_i x \right| \leq |x| \left| w - \sum_{i \in S} w_i \right| < 2\varepsilon |x|$

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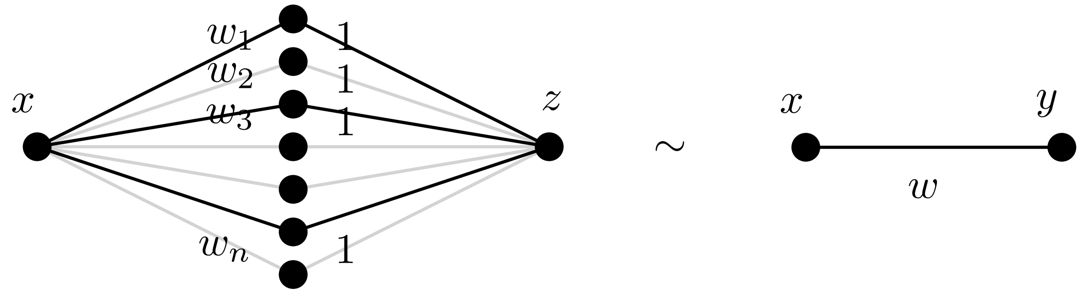

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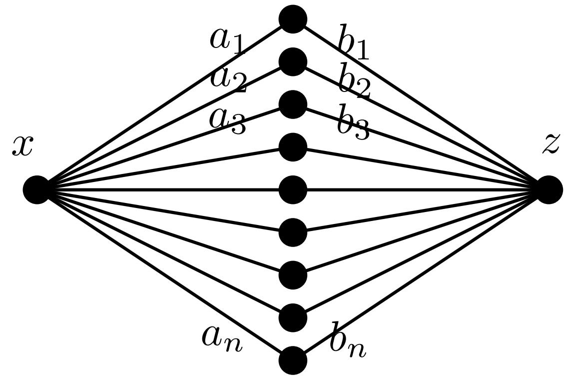
- Completely random initialization + ReLU (non-linearity):



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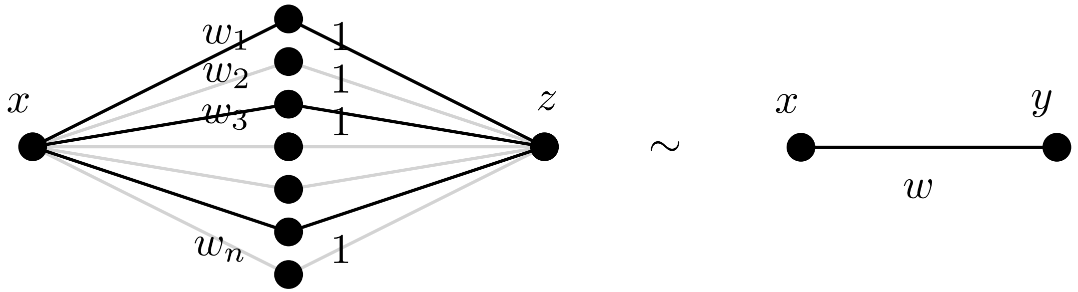

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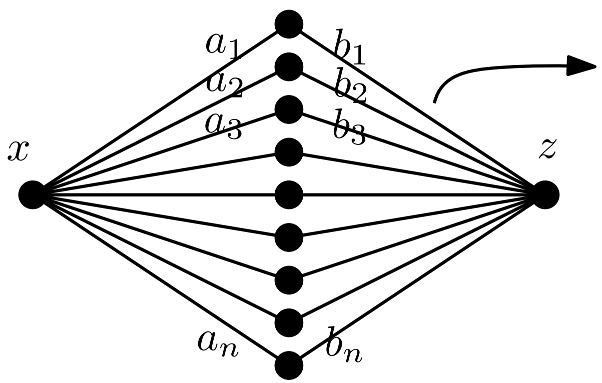
- Completely random initialization + ReLU (non-linearity):

  how to deal with non-linearity?



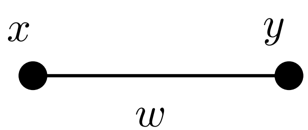
$$\left|wx - \sum_{i \in S} b_i \sigma(a_i x)\right|$$

Exploiting properties of the ReLU

- Completely random initialization + ReLU (non-linearity)

  Property of ReLU: $wx = \sigma(wx) - \sigma(-wx)$

ReLU:
$\sigma(x) = \max(0, x)$



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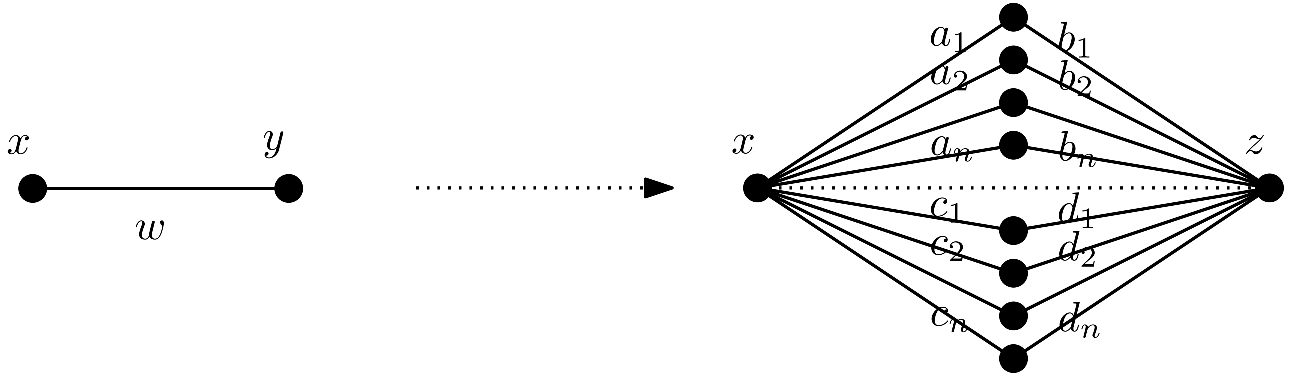

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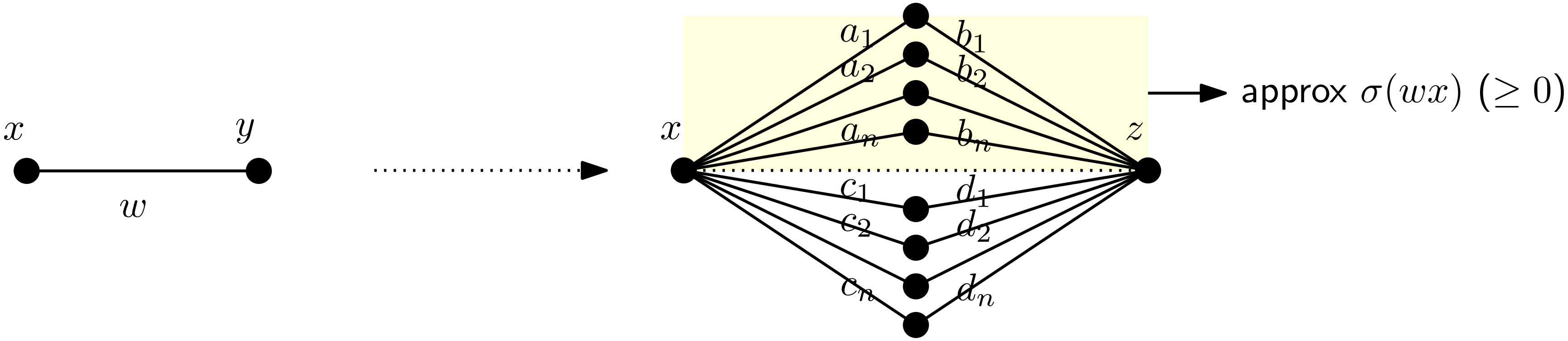

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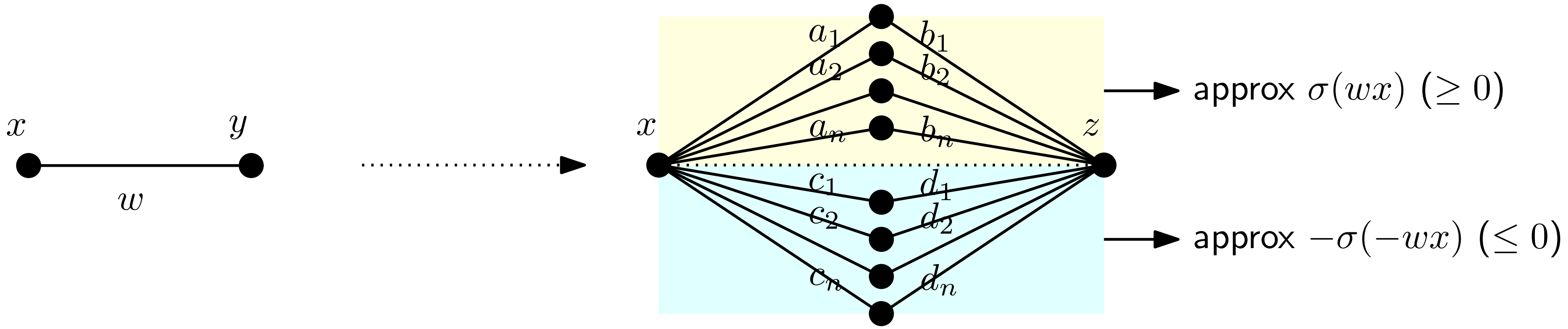

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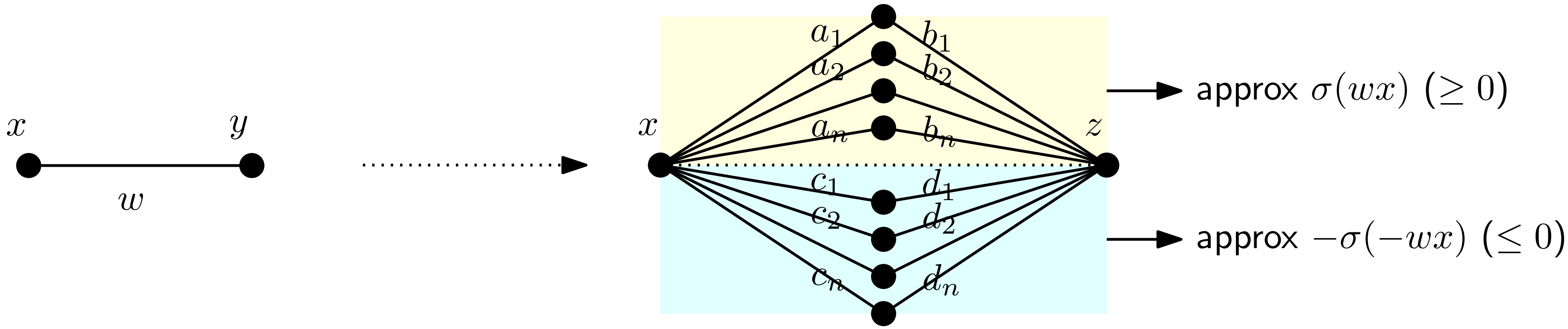
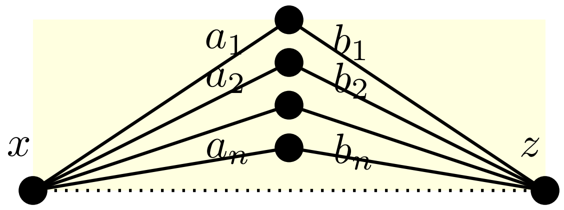


- How? Wlog, assume $w \geq 0$



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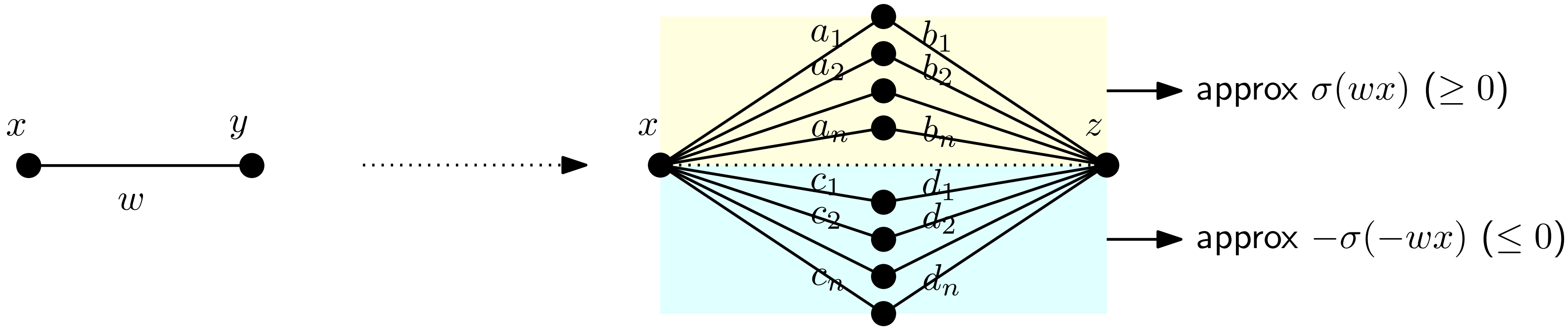

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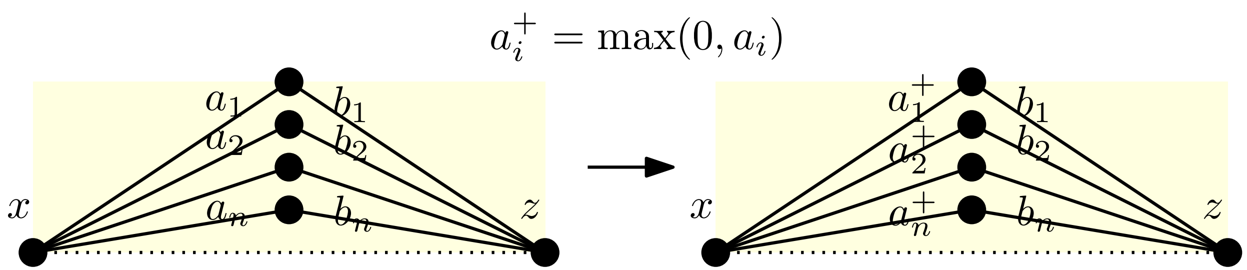

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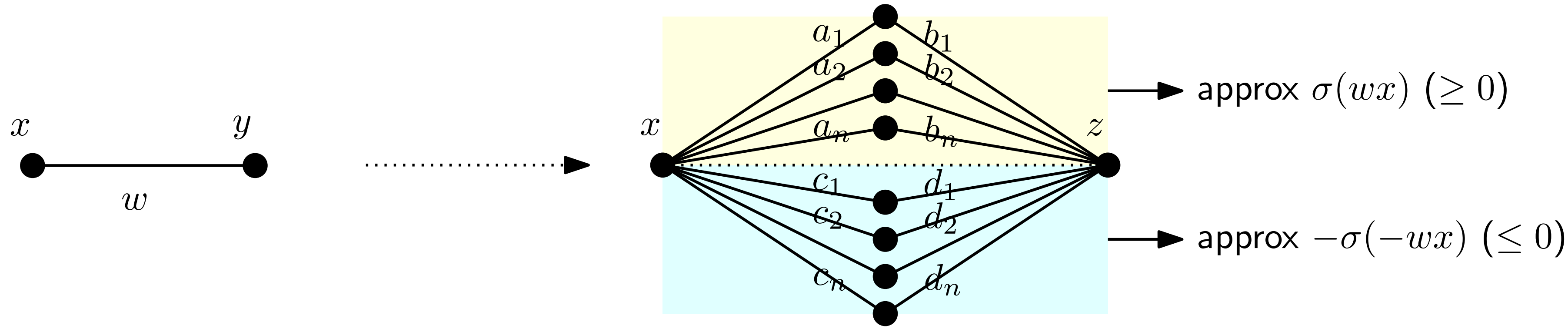
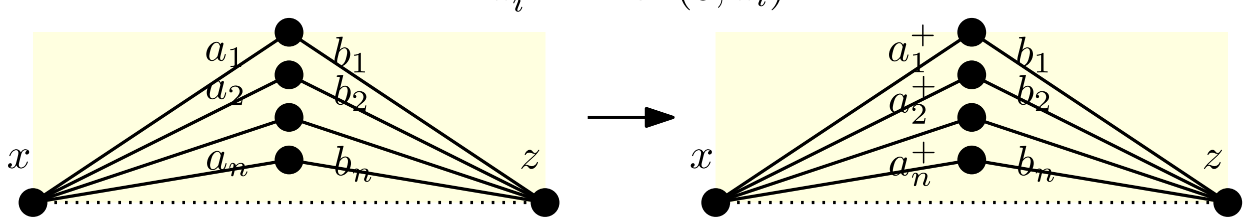


- How? Wlog, assume $w \geq 0$

$a_i^+ = \max(0, a_i)$



$$\left|\sigma(wx) - \sum_{i\in[n]} b_i \sigma(a_i x)\right| = \left|\sigma(wx) - \sum_{i\in[n]} b_i a_i^+ \sigma(x)\right|$$

Exploiting properties of the ReLU

- Completely random initialization + ReLU (non-linearity)

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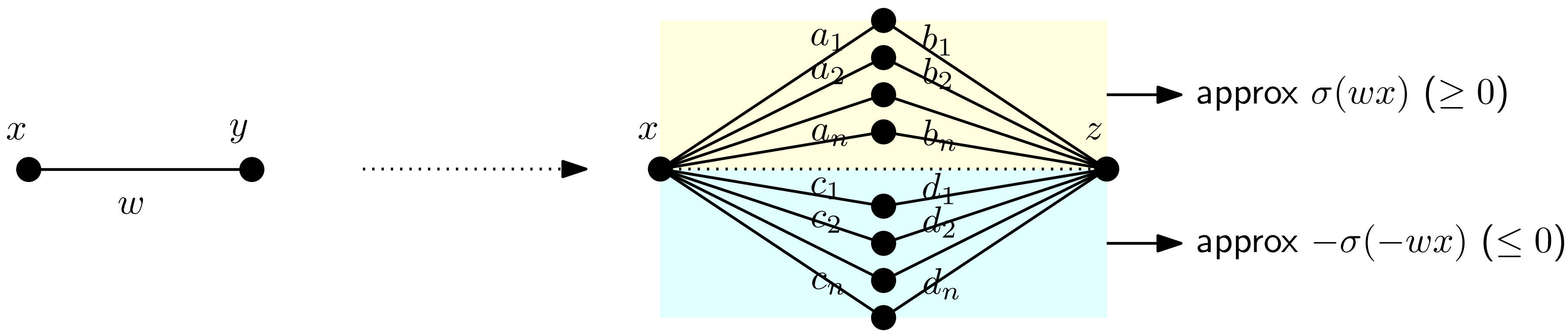

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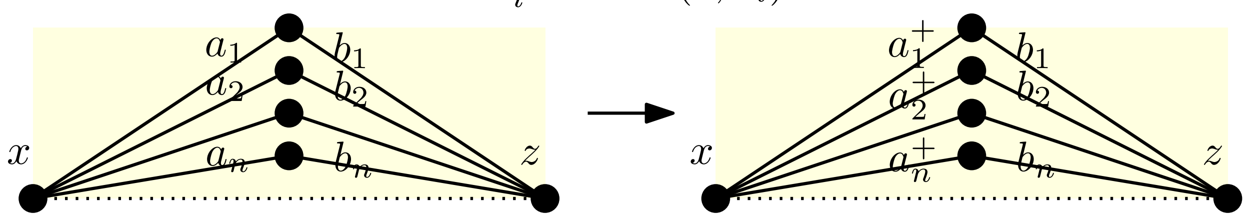

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if $x \leq 0$, easy

Exploiting properties of the ReLU

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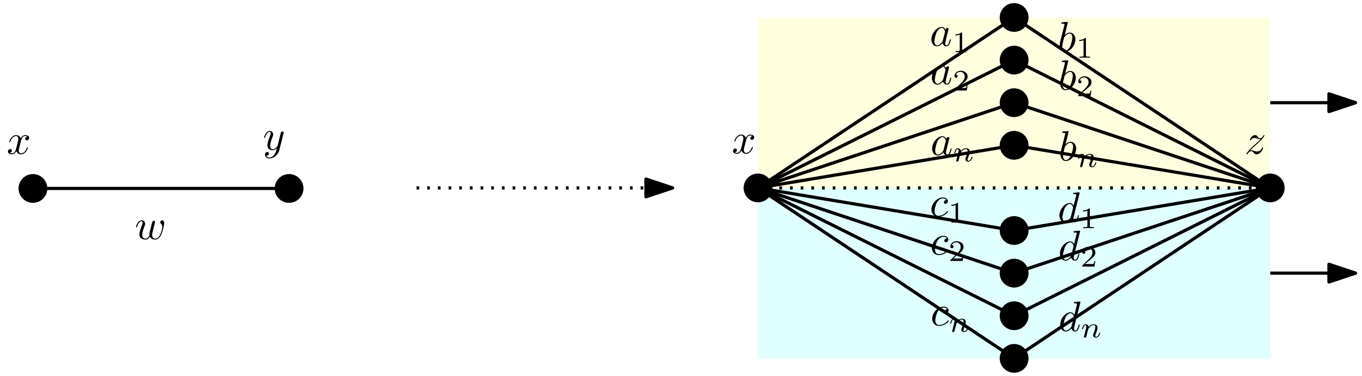

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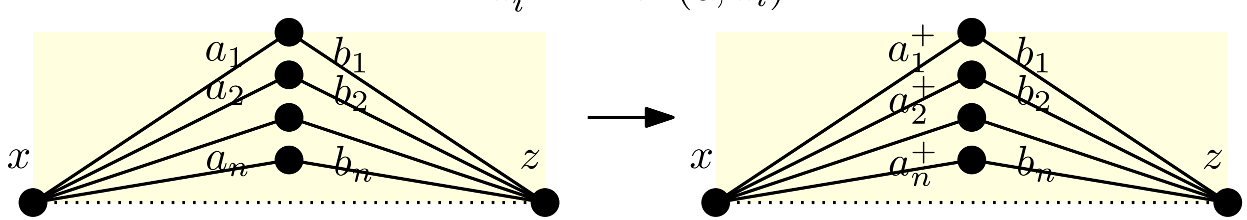

$$\left|\sigma(wx) - \sum_{i\in[n]} b_i\sigma(a_i x)\right| = \left|\sigma(wx) - \sum_{i\in[n]} b_i a_i^+ \sigma(x)\right| = x\left|w - \sum_{i\in[n]} b_i a_i^+\right|$$

if $x \leq 0$, easy

if $x > 0$, then RSS holds!

Exploiting properties of the ReLU

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- Completely random initialization + ReLU (non-linearity)

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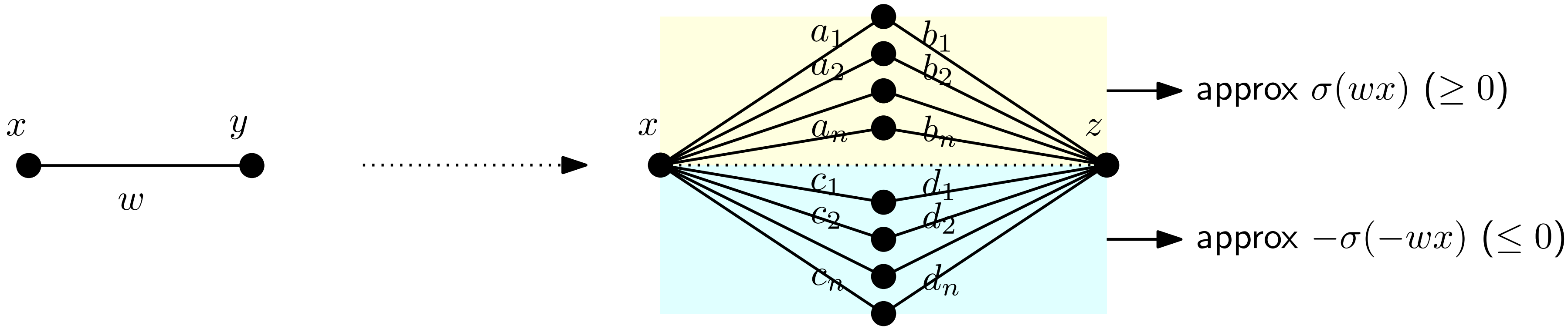

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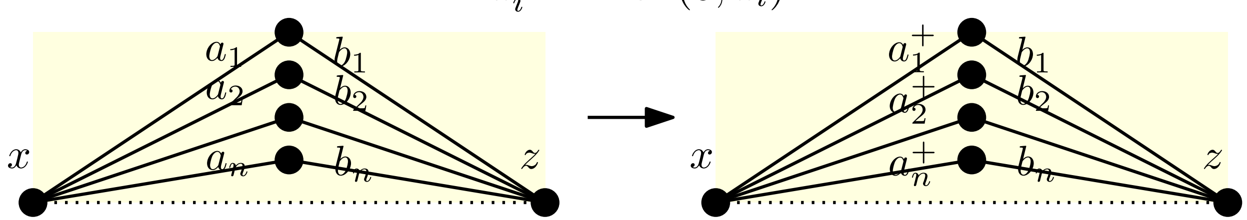

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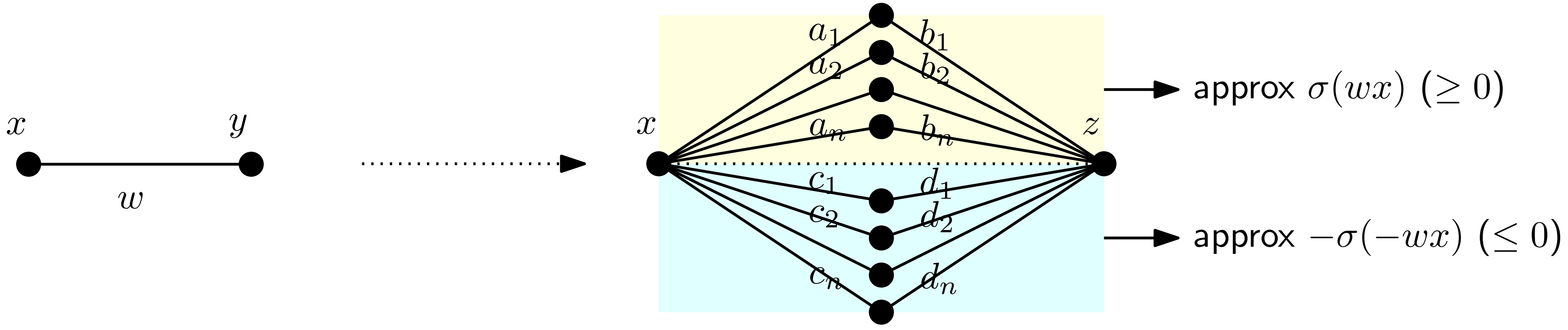
if $x > 0$, then RSS holds!

$n \geq C \log 1/\varepsilon$

Putting everything together



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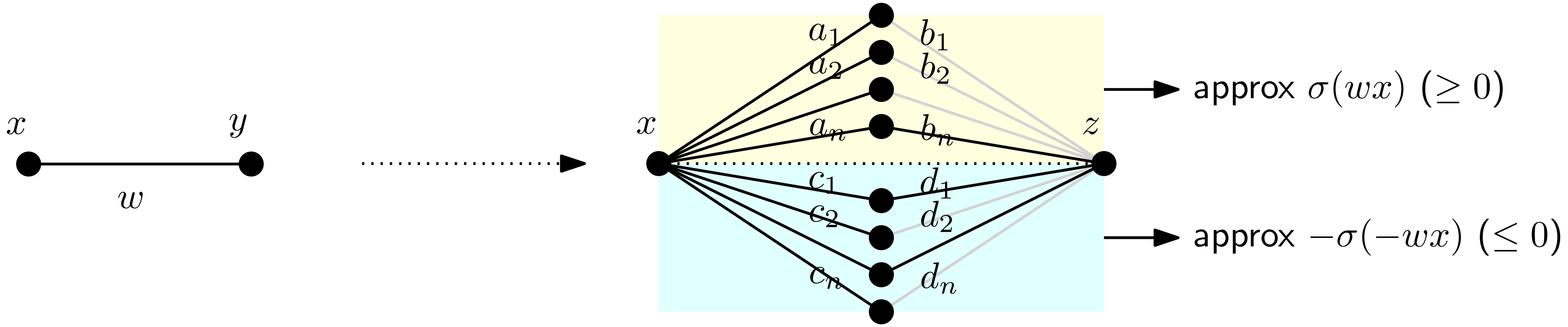

prune only the right layer: reuse the left layer

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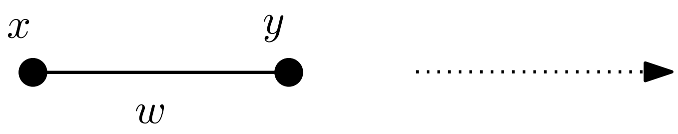

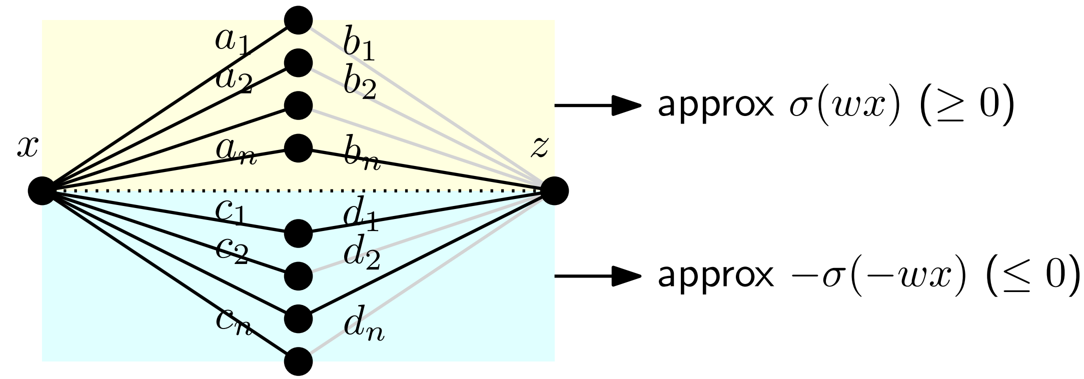

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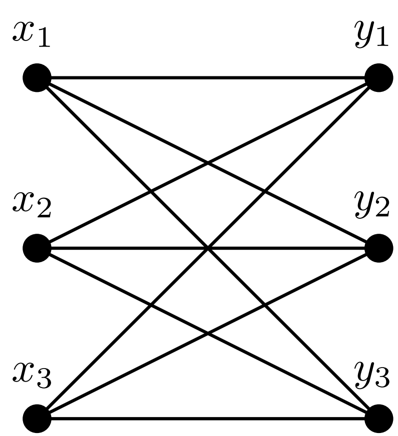

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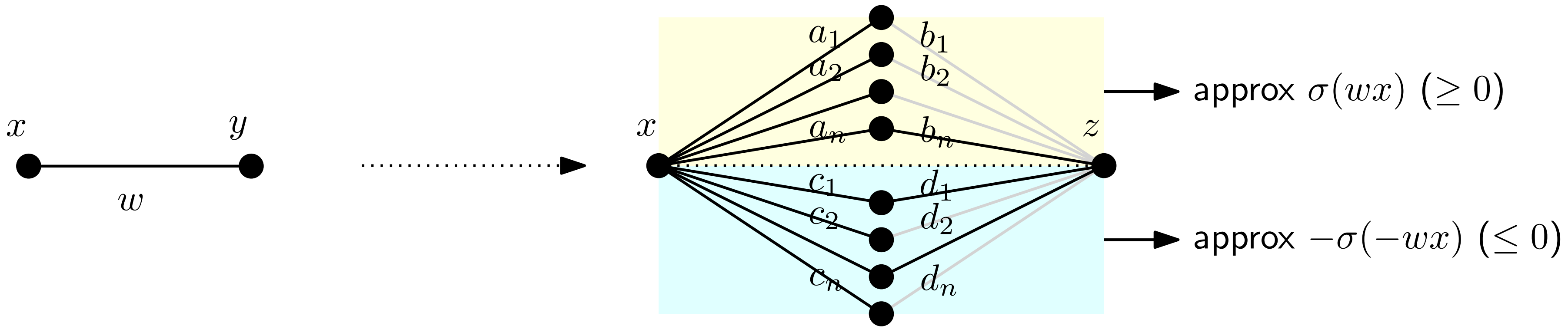

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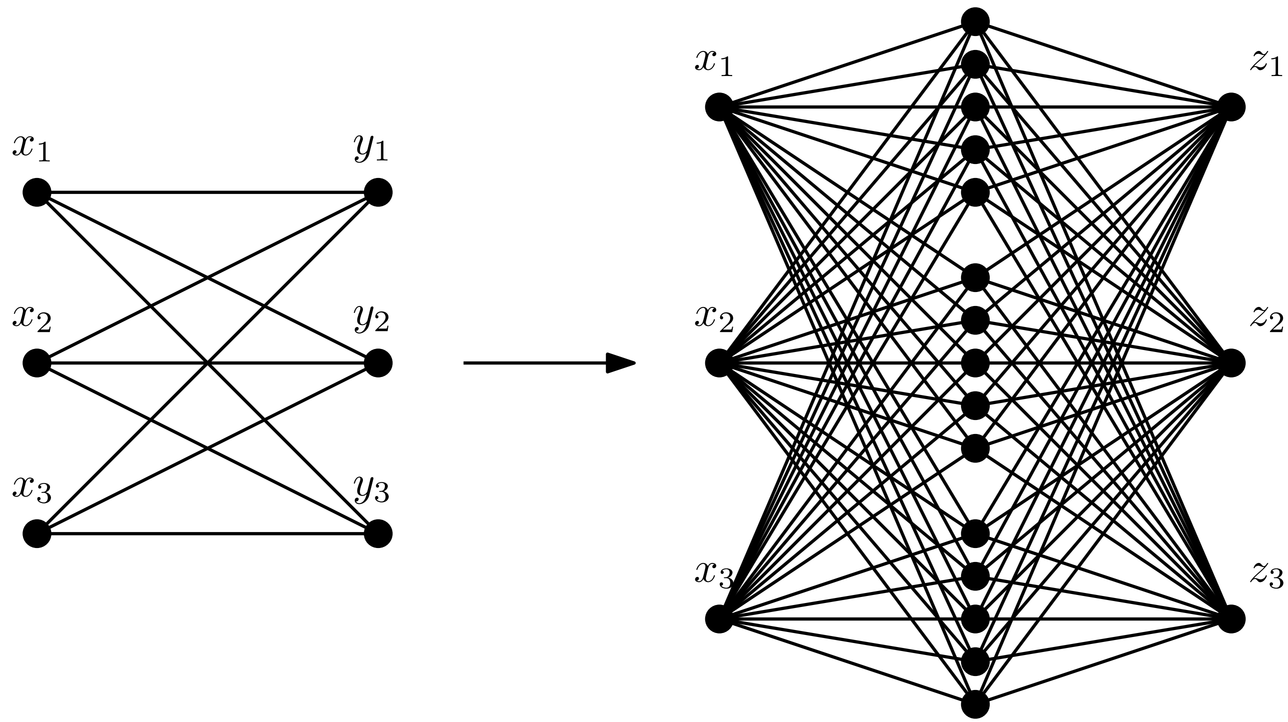

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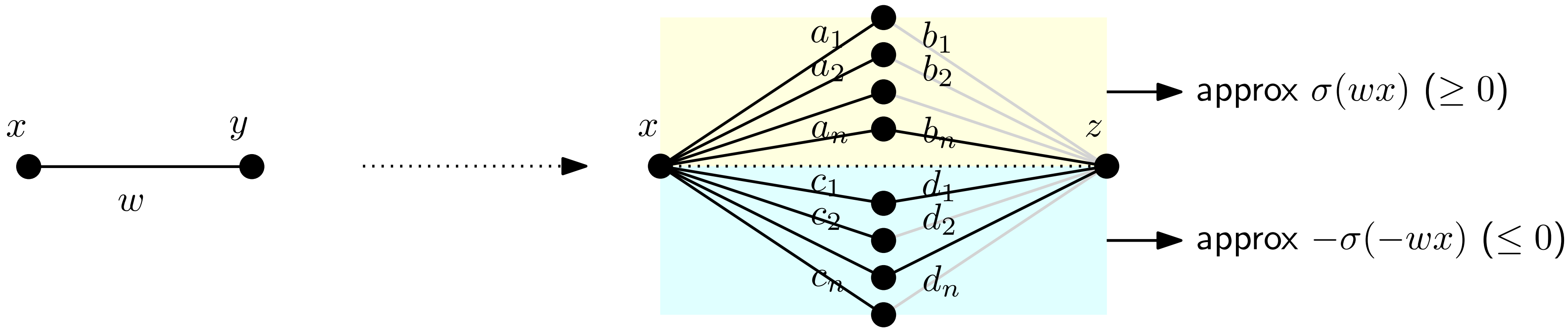

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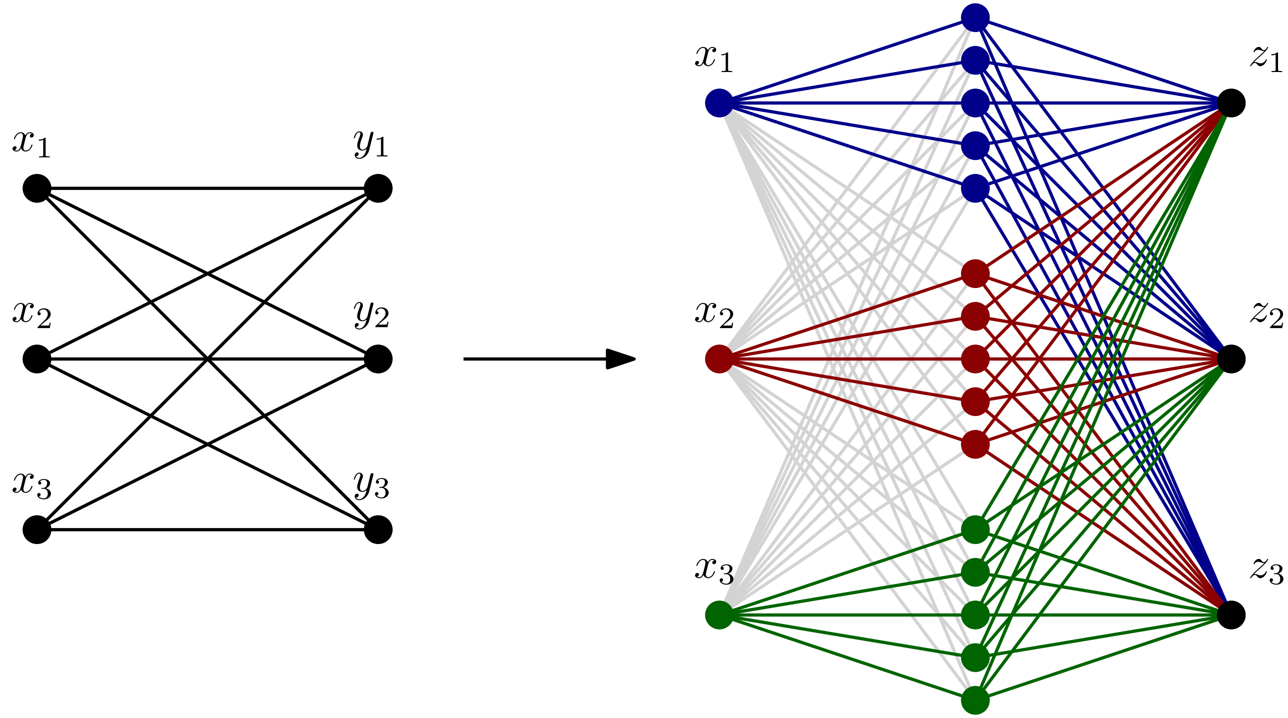

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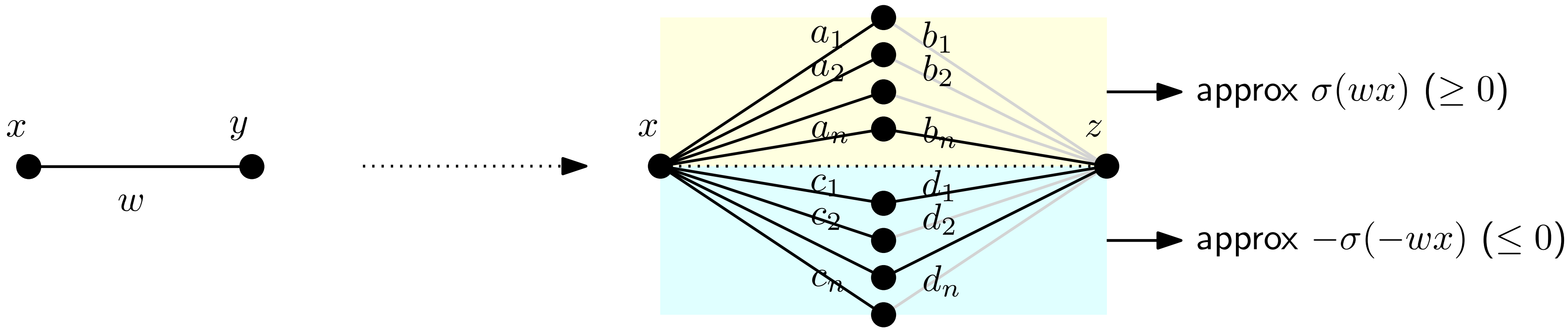

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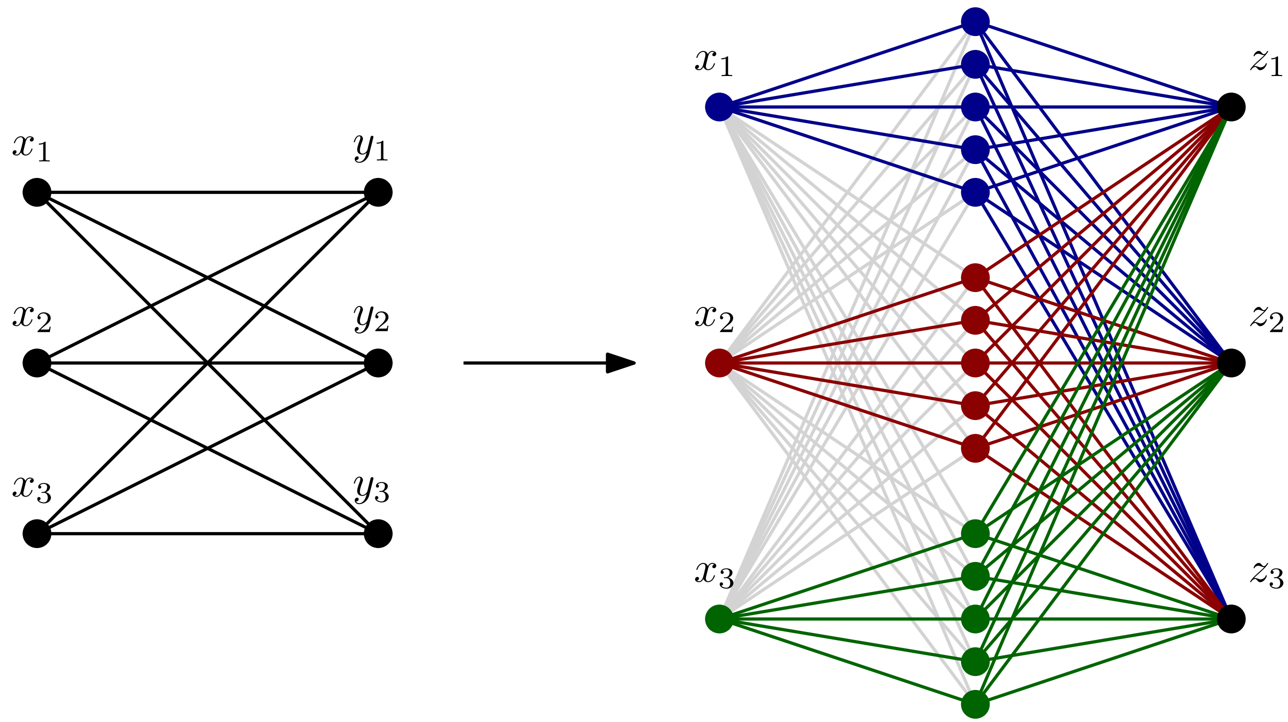
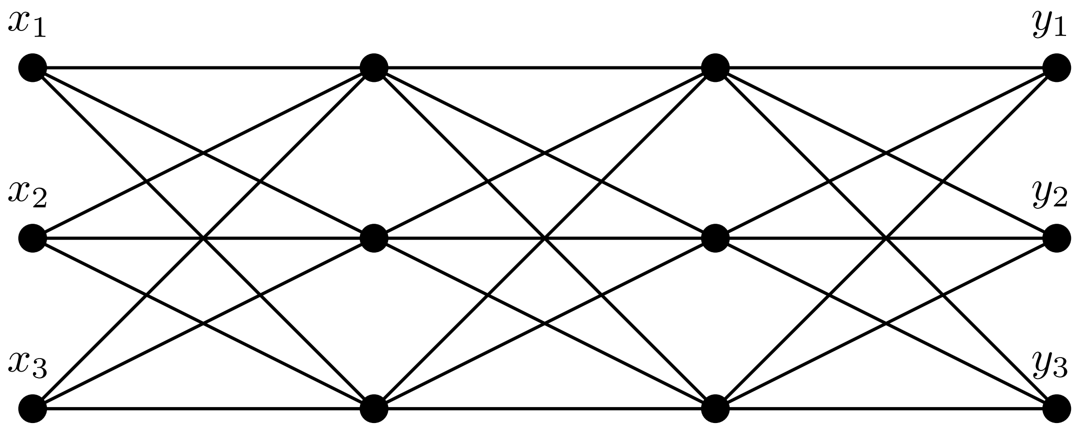


$n \geq C \log d^2/\varepsilon$

d = width original layer

More layers together



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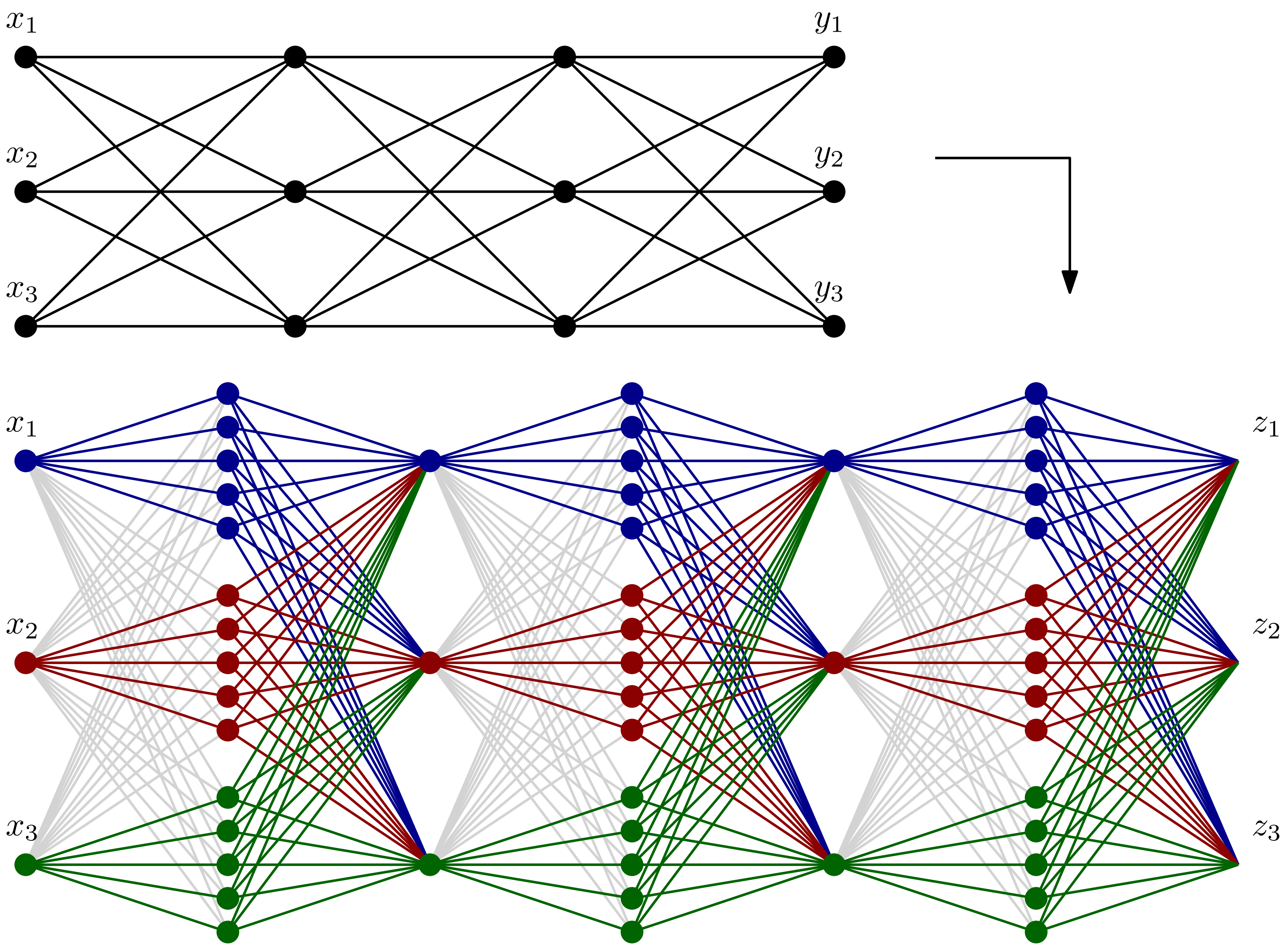
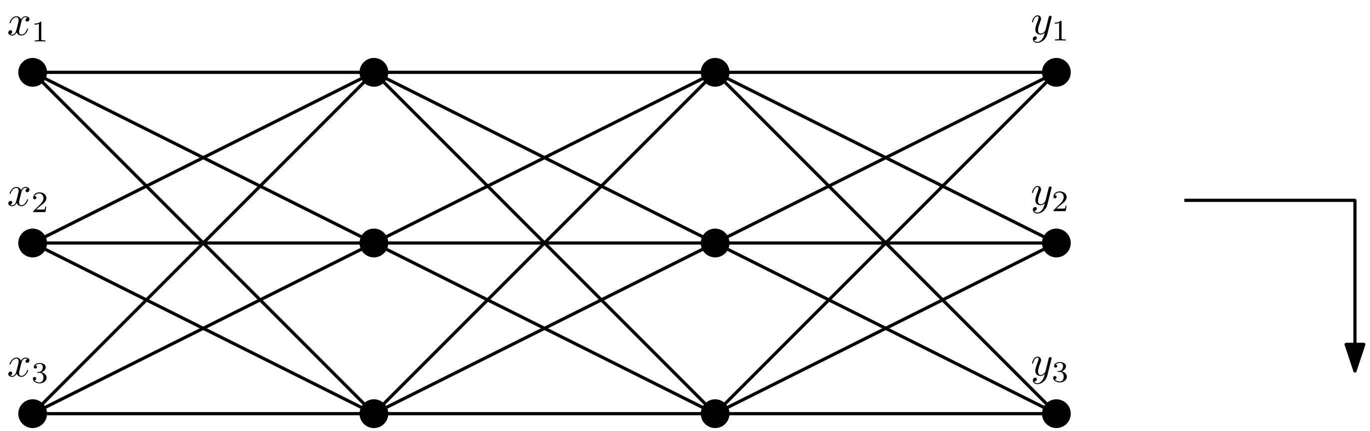


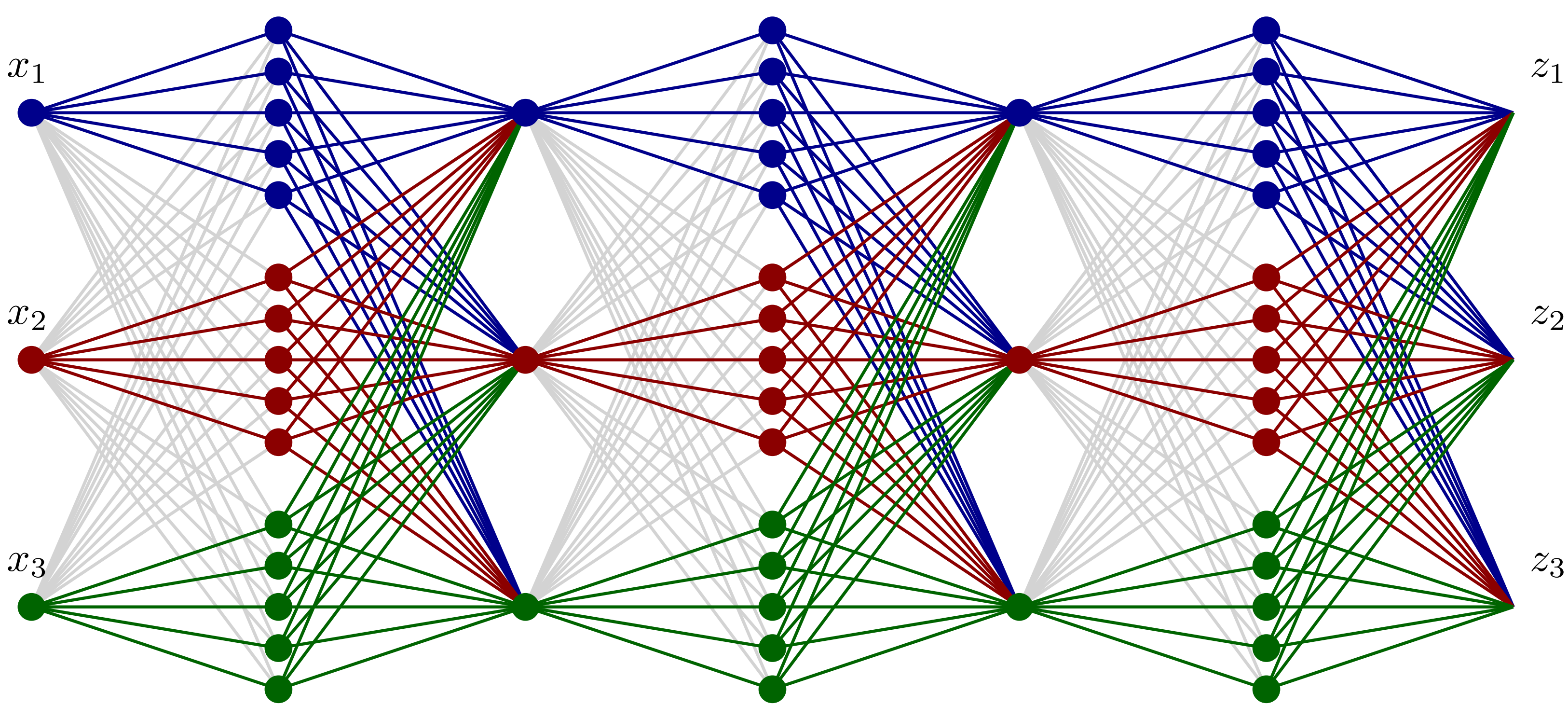
More layers together



$n \geq C \log \ell d^2/\varepsilon$

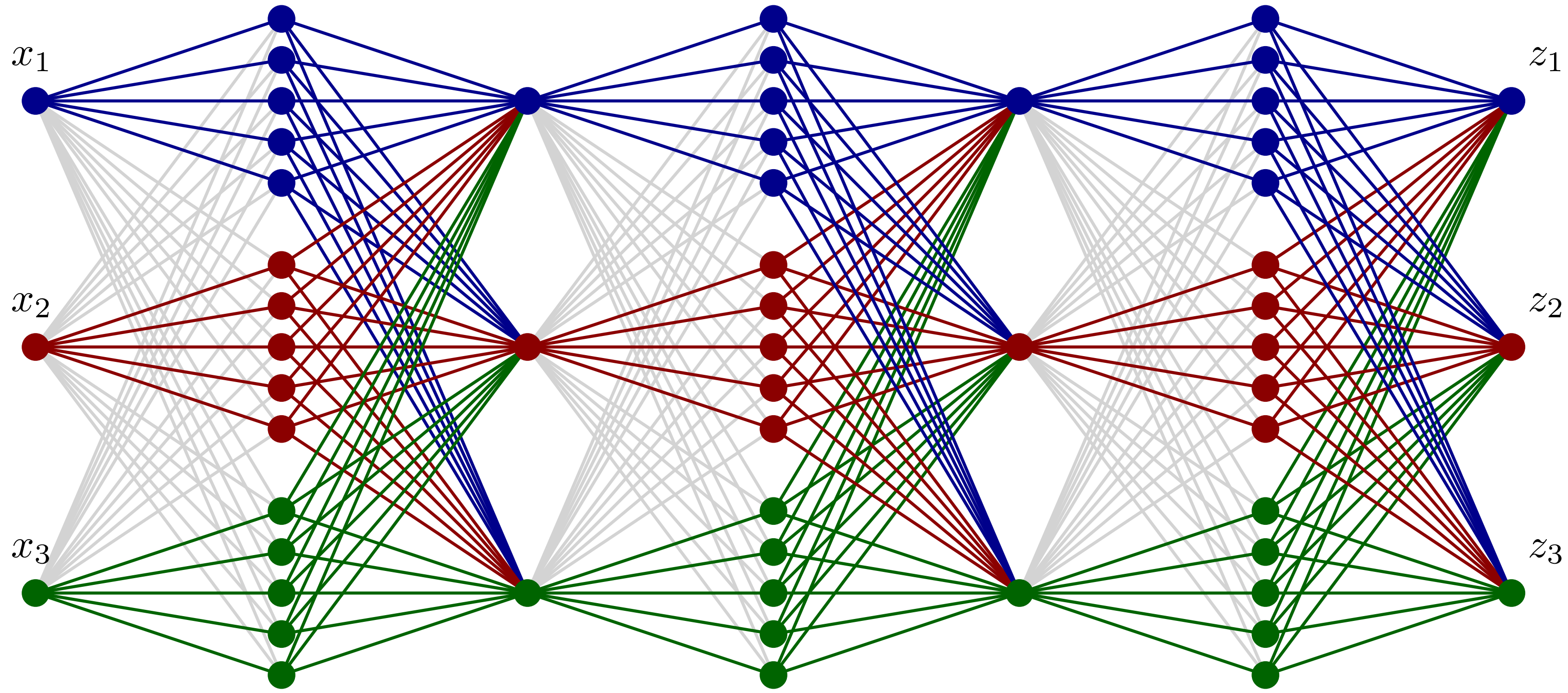
$\ell = \#$ original layers

$\implies \|\mathbf{y} - \mathbf{z}\| \leq 2\varepsilon$



Issues with Unstructured pruning

- Removed edges can be everywhere



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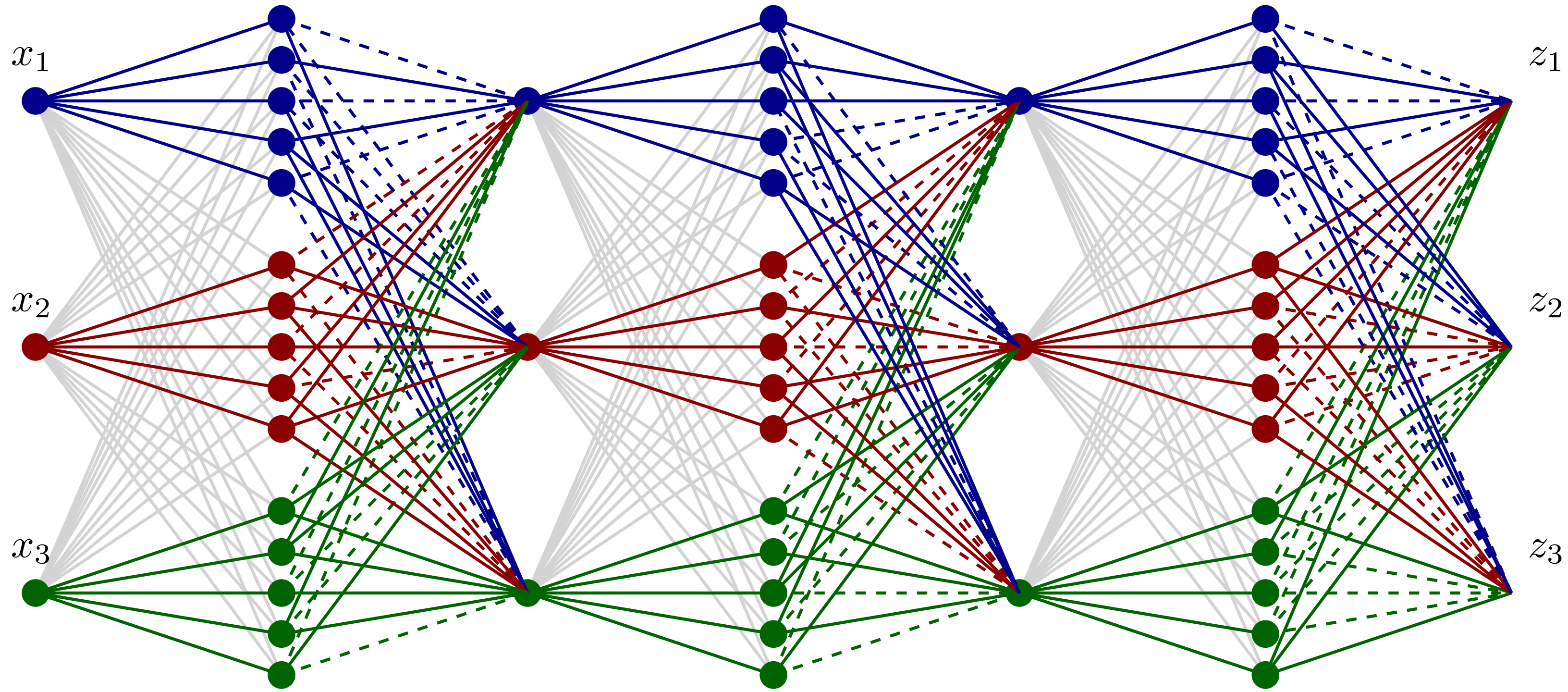

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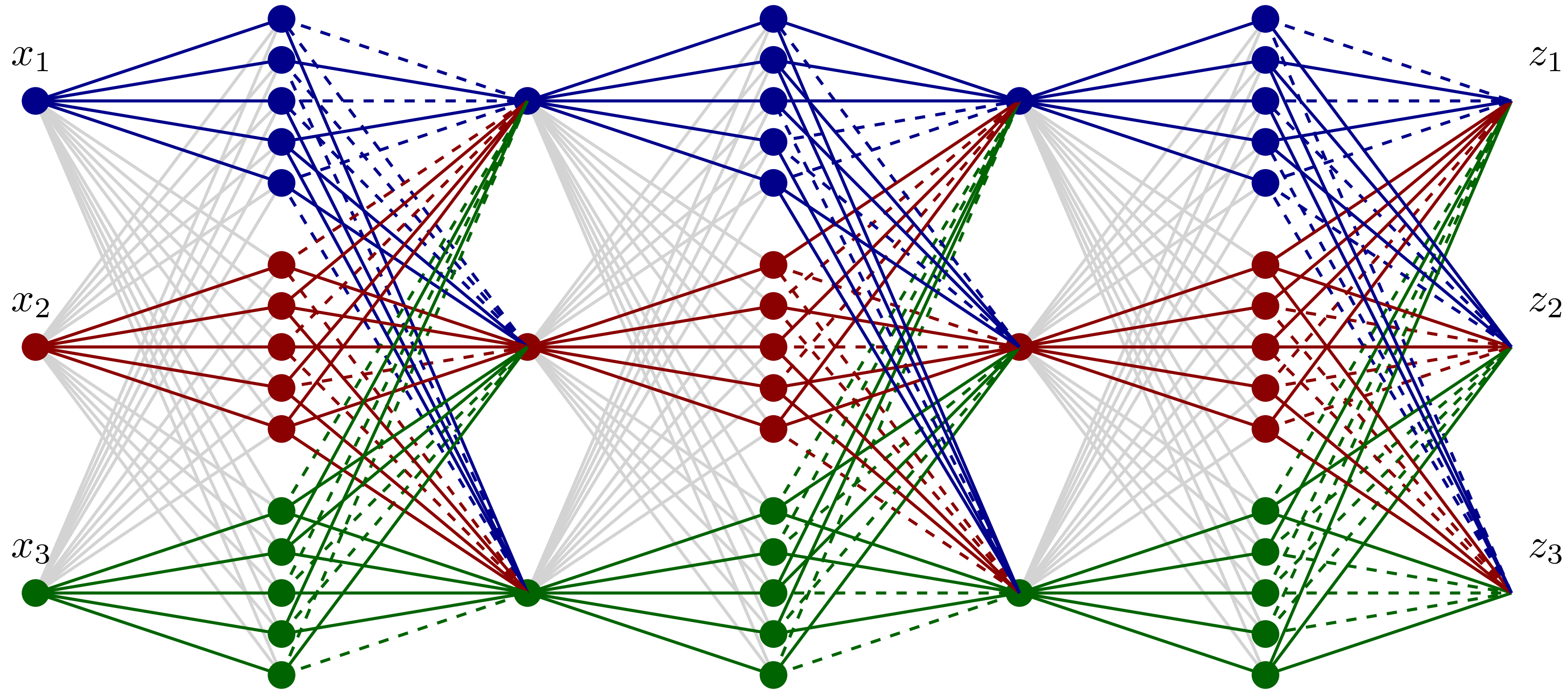

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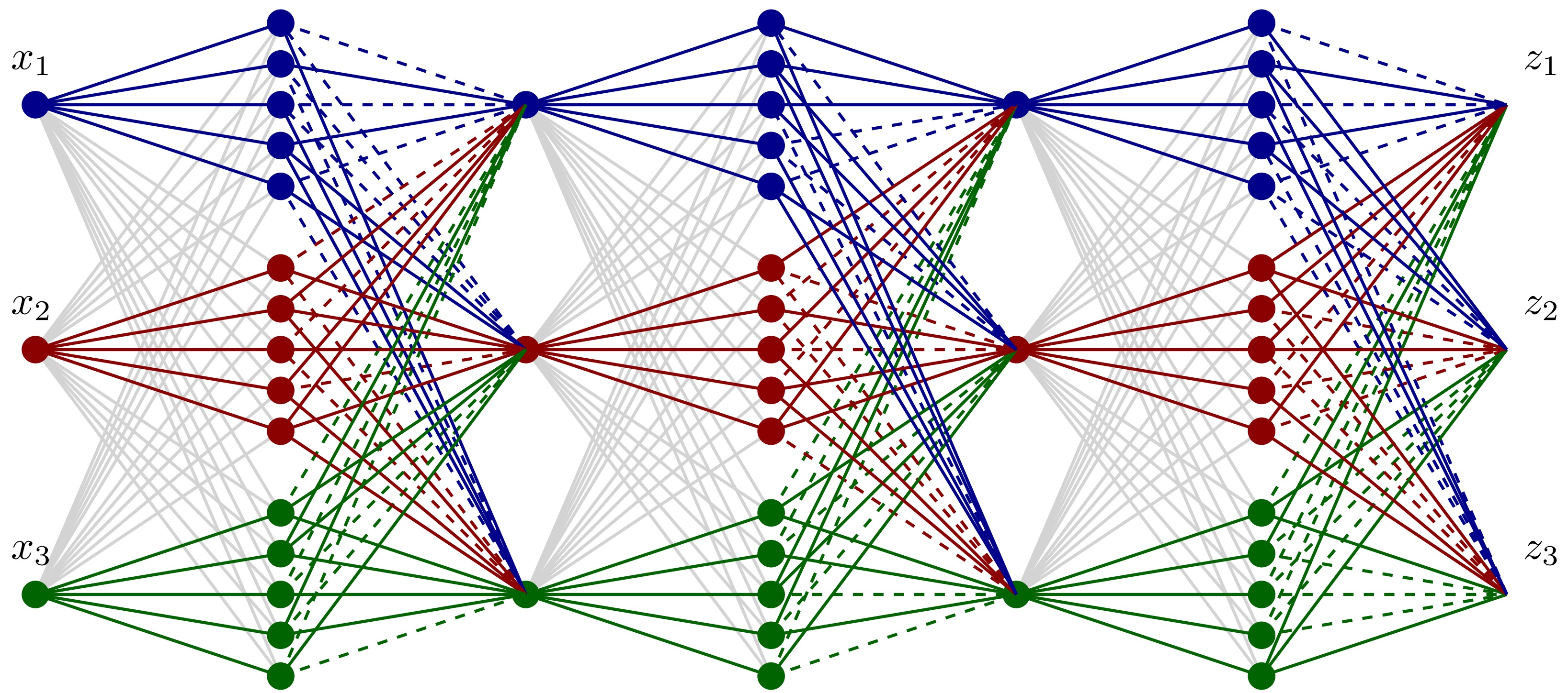

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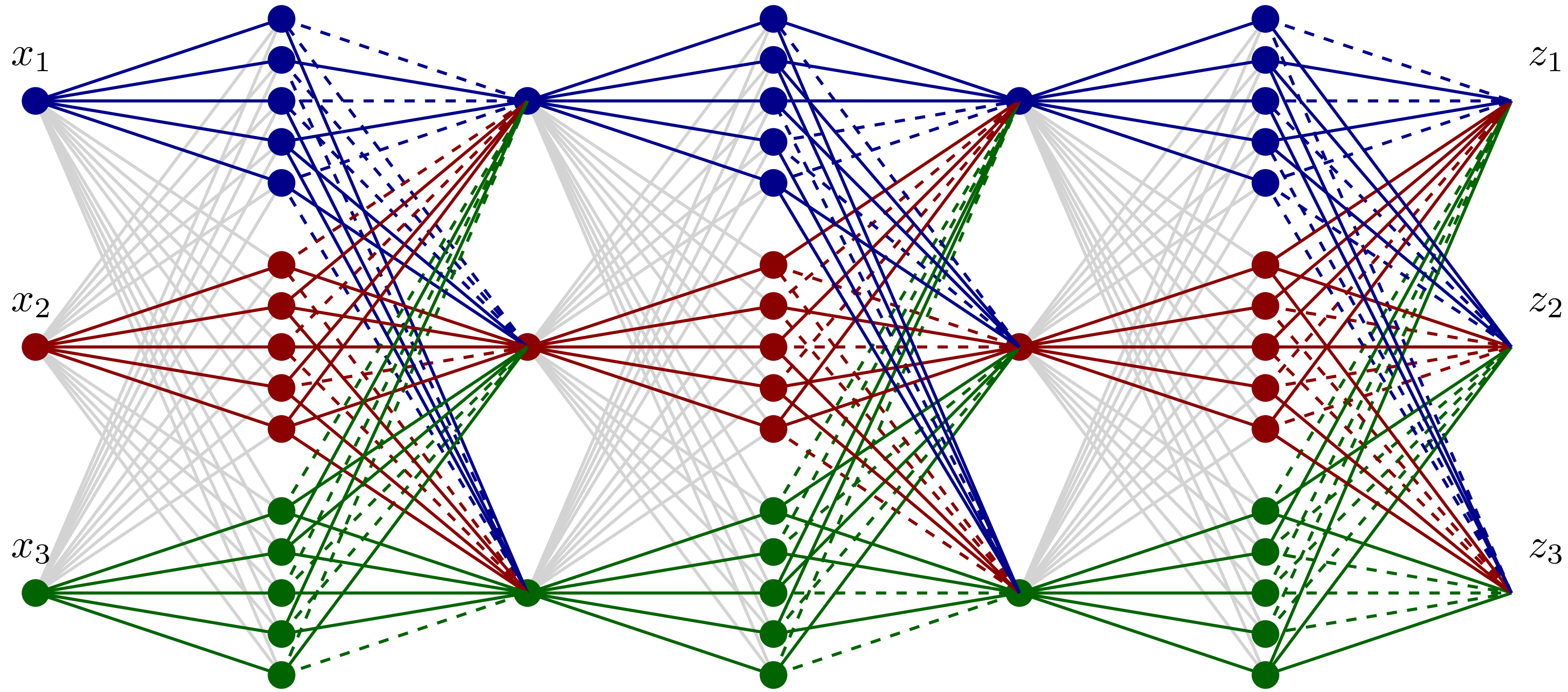
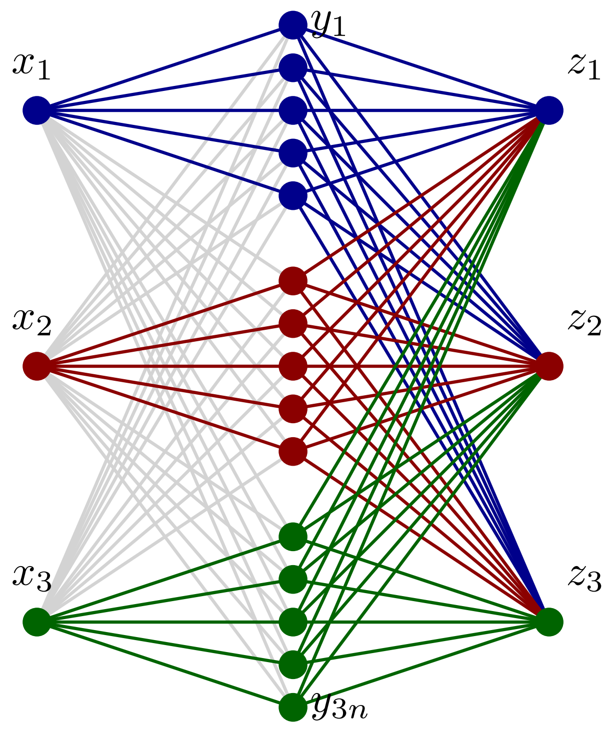


- No structure usually implies slower processes
  - difficulty encoding unstructured sparsity
  - accessing data is more time consuming than processing

Structured pruning



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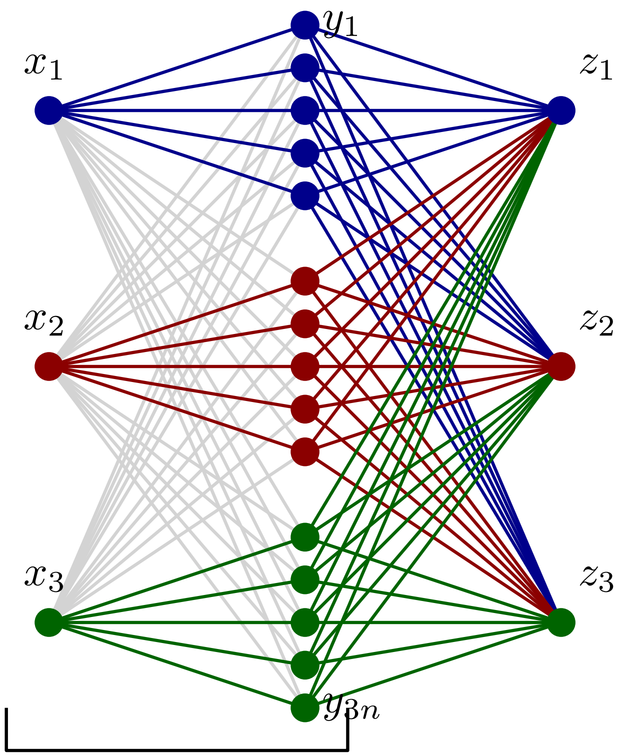

$$\begin{bmatrix} w_{1,1} & 0 & 0 \\ \vdots & \vdots & \vdots \\ w_{n,1} & 0 & 0 \\ 0 & w_{1,2} & 0 \\ \vdots & \vdots & \vdots \\ 0 & w_{n,2} & 0 \\ 0 & 0 & w_{1,3} \\ \vdots & \vdots & \vdots \\ 0 & 0 & w_{n,3} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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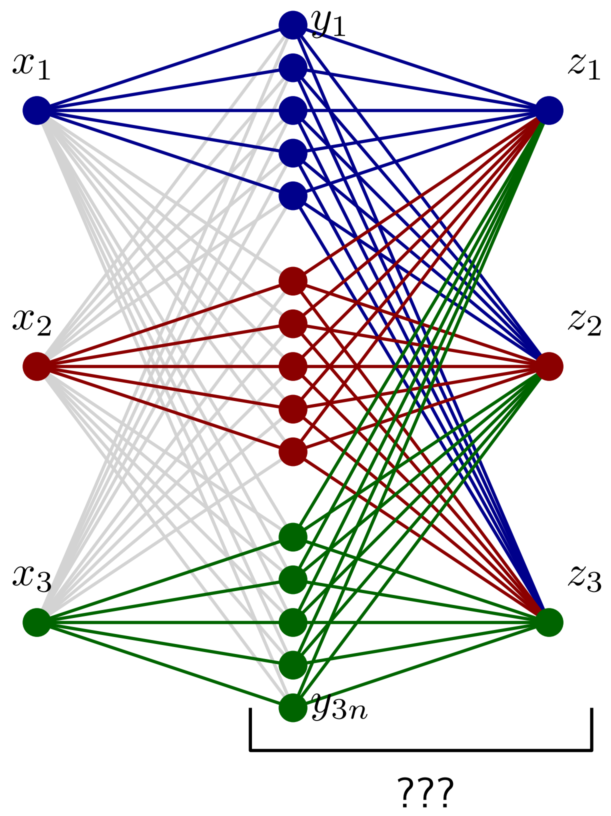

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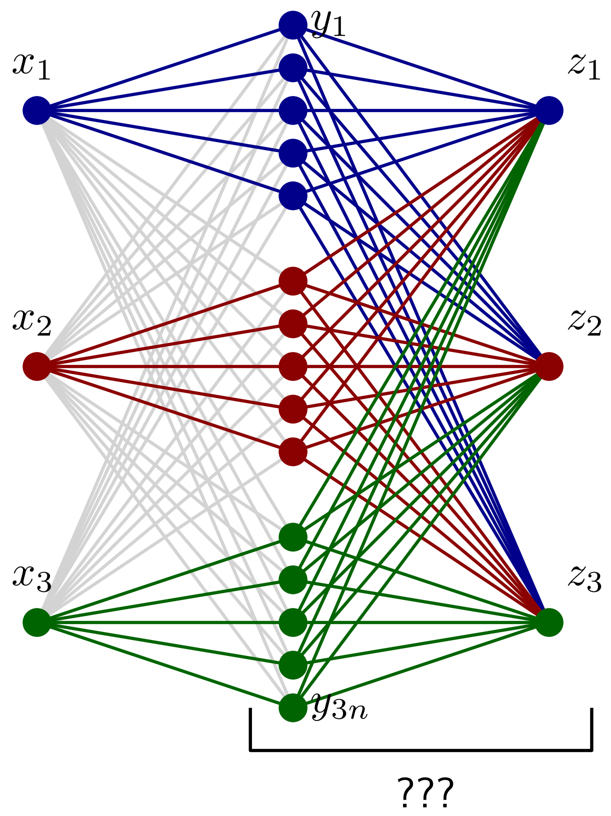
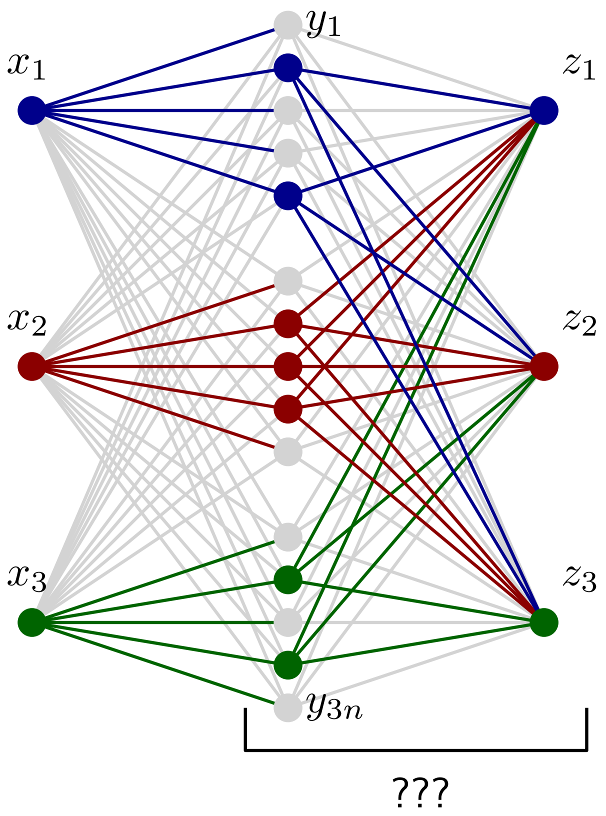


- Removing entire neurons from the middle layer!

Structured pruning



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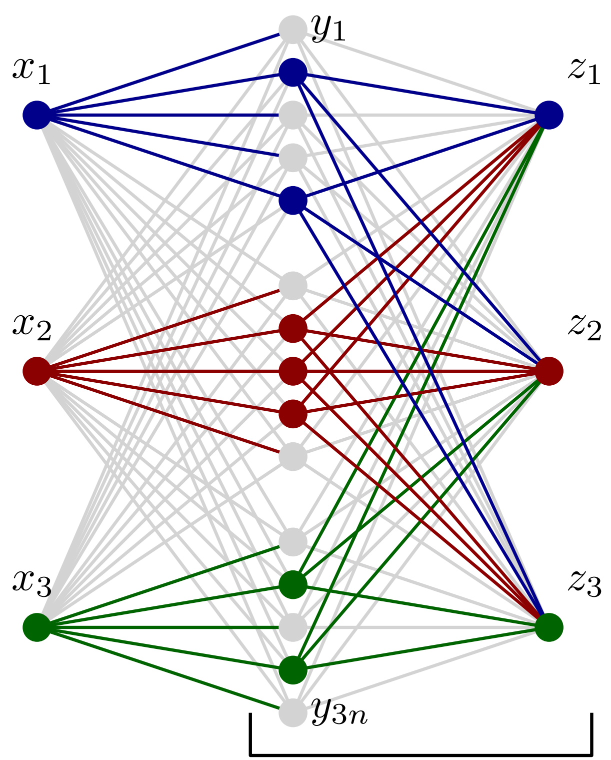
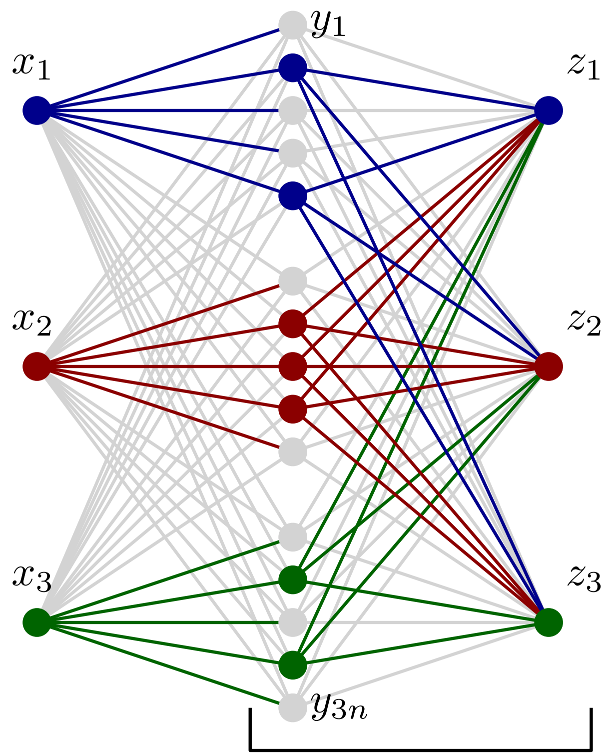


- Removing entire neurons from the middle layer!
  - removes columns!

$$\begin{bmatrix} 0 & v_{1,2} & 0 & \dots & 0 & v_{i,1} & 0 & \dots \\ 0 & v_{2,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \\ 0 & v_{3,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{3n} \end{bmatrix}$$

Structured pruning



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  - leads to exponential bounds

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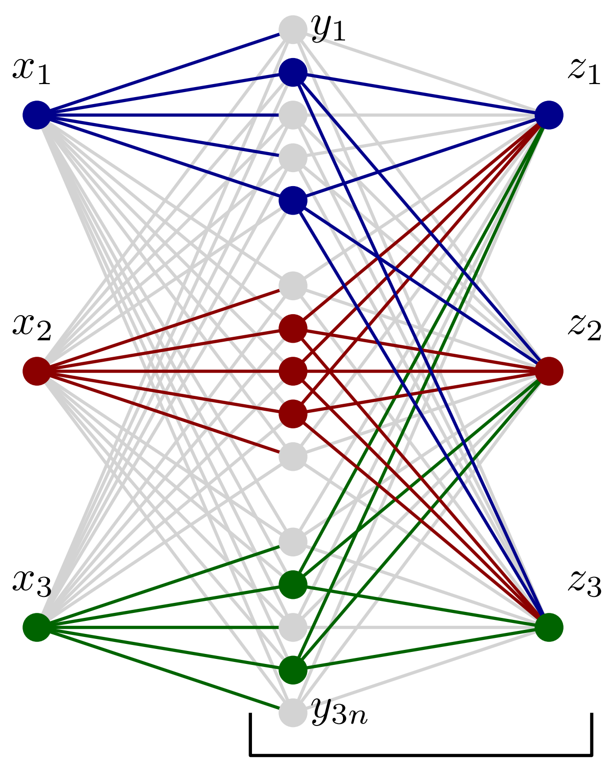

- Removing entire neurons from the middle layer!
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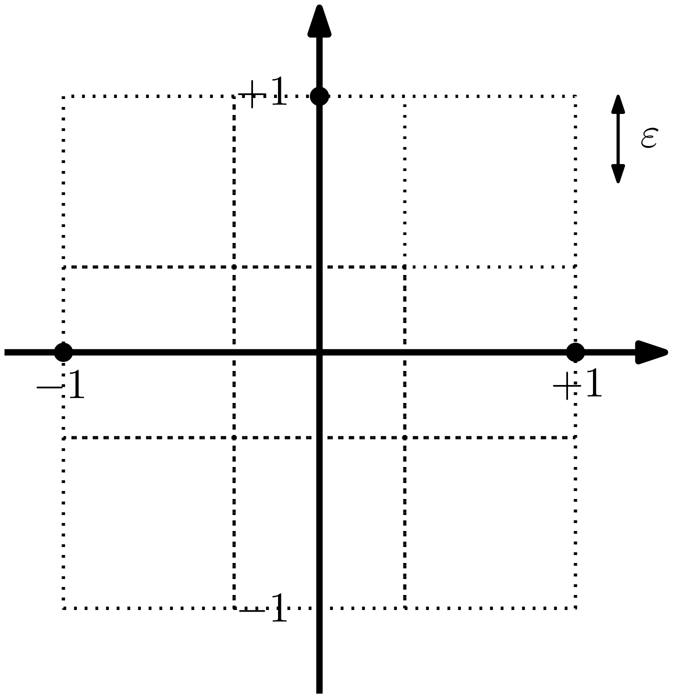
- The one-dimensional RSS result does not work
  - leads to exponential bounds
- A multidimensional RSS result is required

$$\begin{bmatrix} 0 & v_{1,2} & 0 & \dots & 0 & v_{i,1} & 0 & \dots \\ 0 & v_{2,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \\ 0 & v_{3,2} & 0 & \dots & 0 & v_{i,2} & 0 & \dots \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{3n} \end{bmatrix}$$

The multidimensional RSS problem

- Natural generalization



The multidimensional RSS problem

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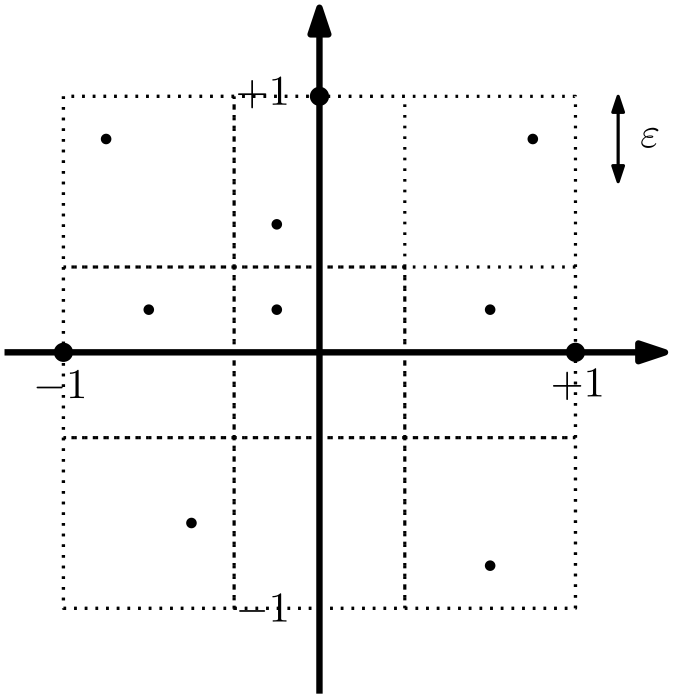

Input:

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The multidimensional RSS problem

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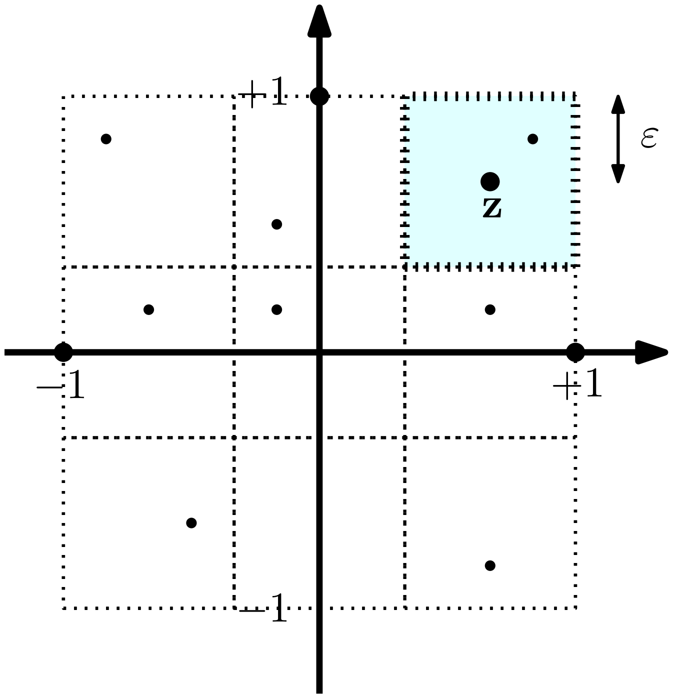

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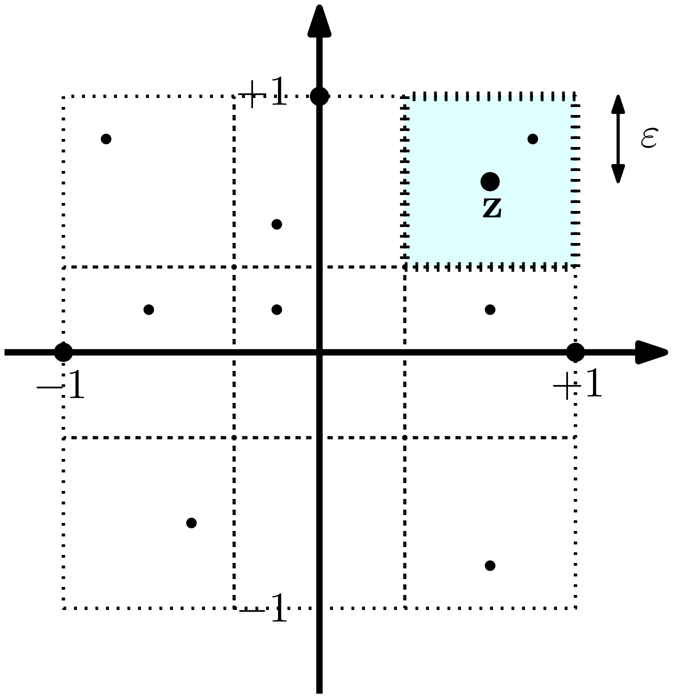

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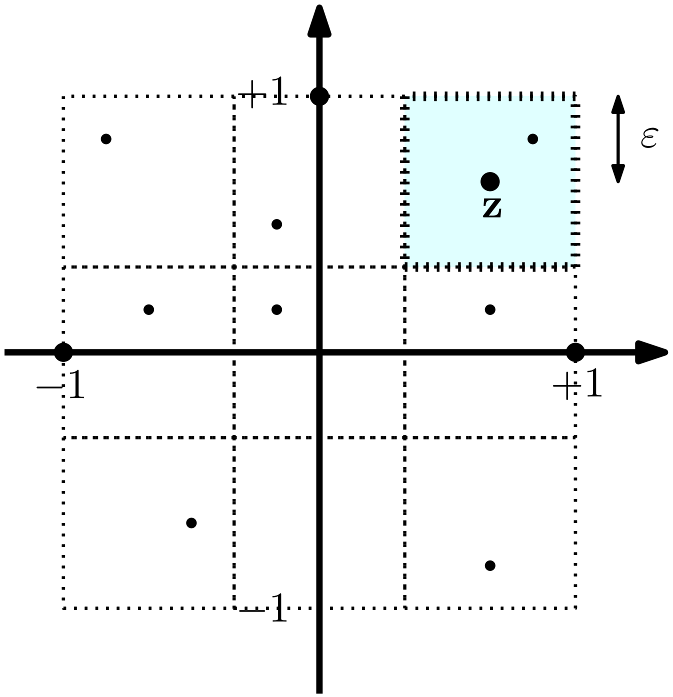

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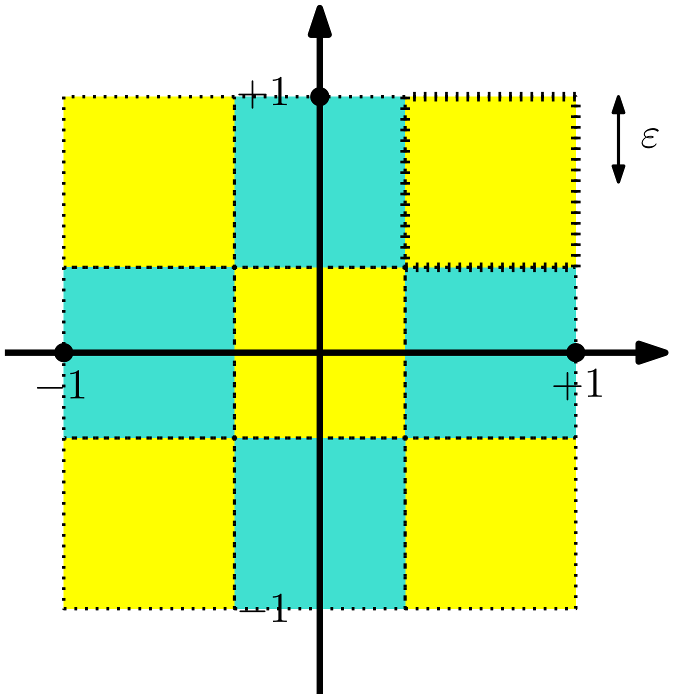
Question:

• Estimate n such that, with high probability, a subset $S \subseteq [n]$ exists with $\|\mathbf{z} - \sum_{i \in S} X_i\|_\infty \leq 2\varepsilon$

MRSS in expectation

- Number of ε-cubes: $1/\varepsilon^d = 2^{d \log 1/\varepsilon}$



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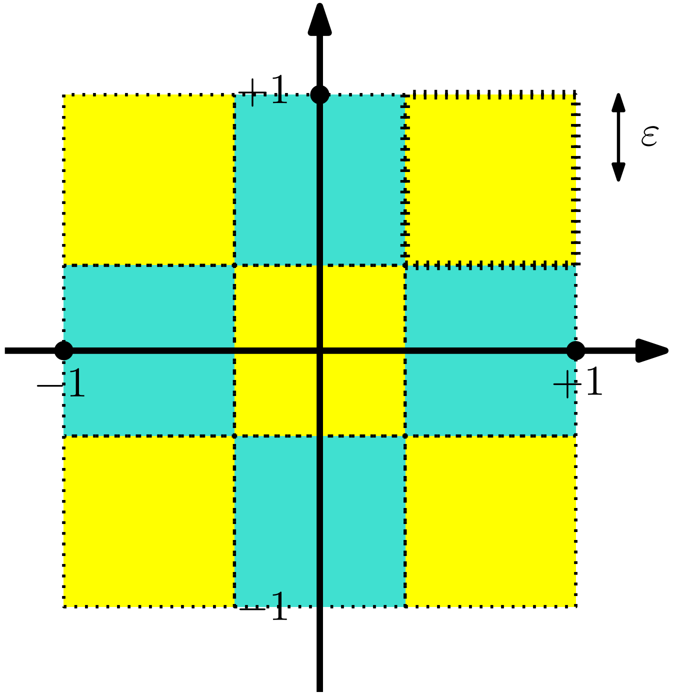

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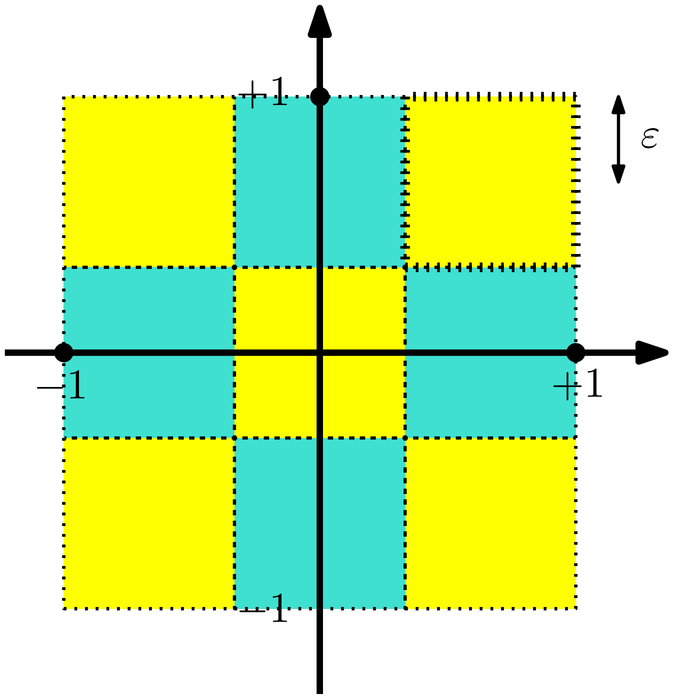

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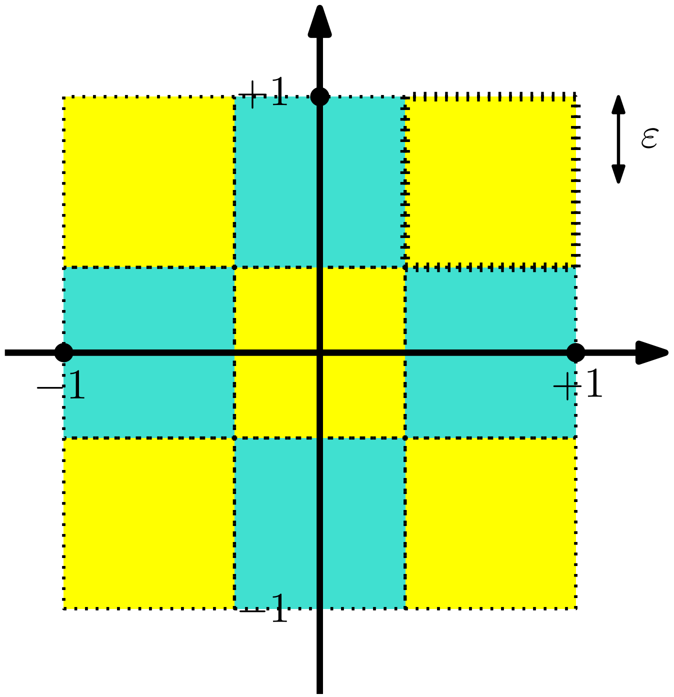
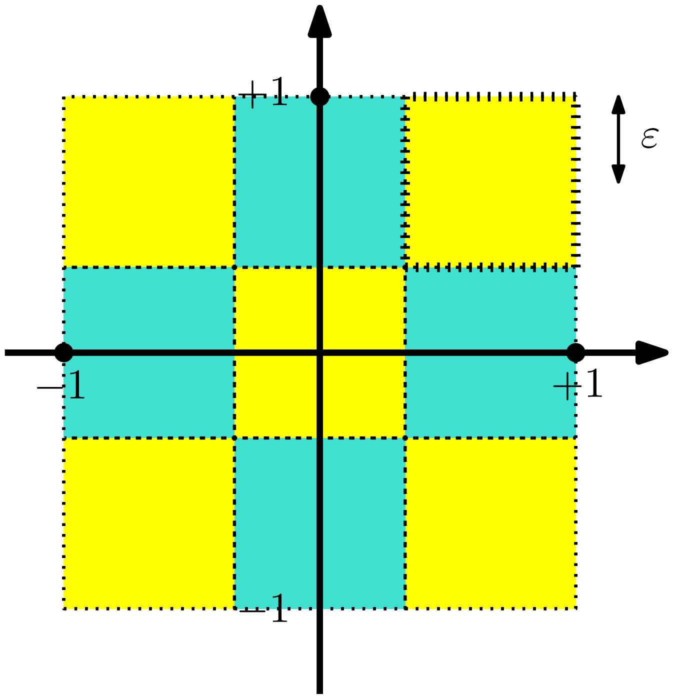


Upper bound

MRSS in expectation



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- If subset size $k = \frac{n}{2}$, possible subsets: $\binom{n}{n/2} \geq 2^{n/2}$

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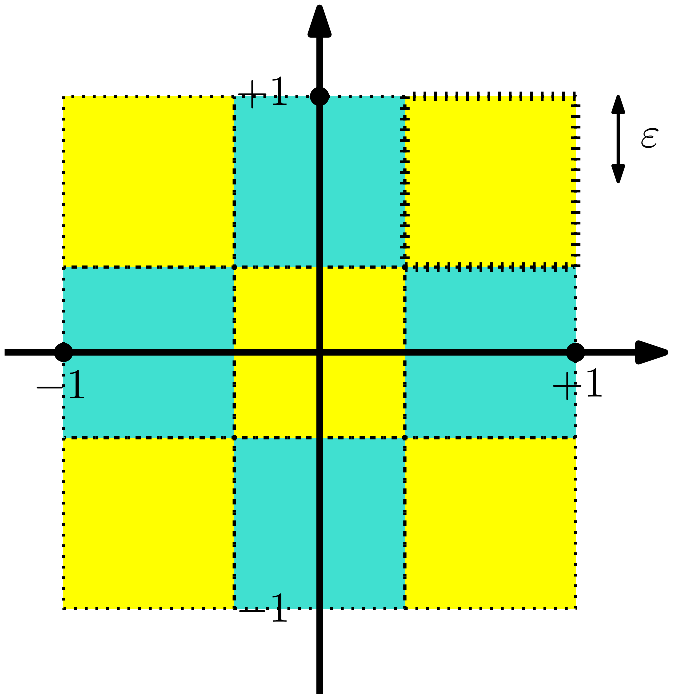

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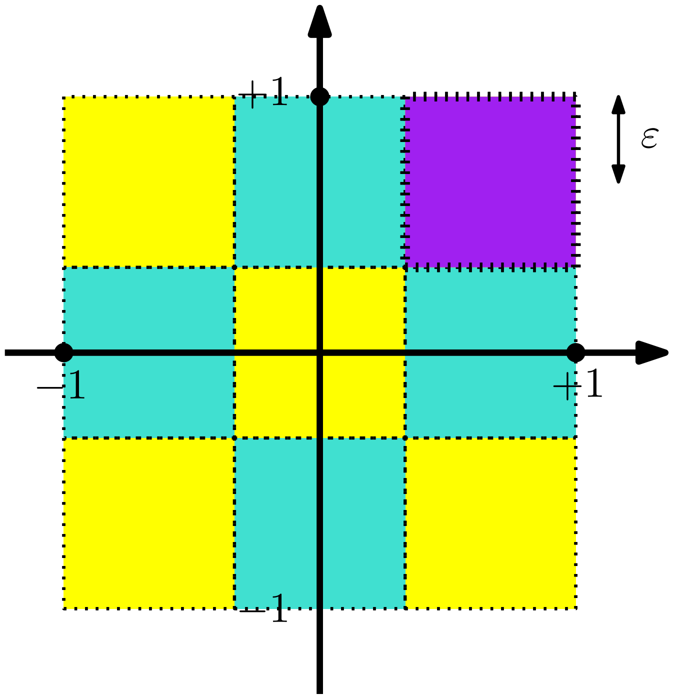

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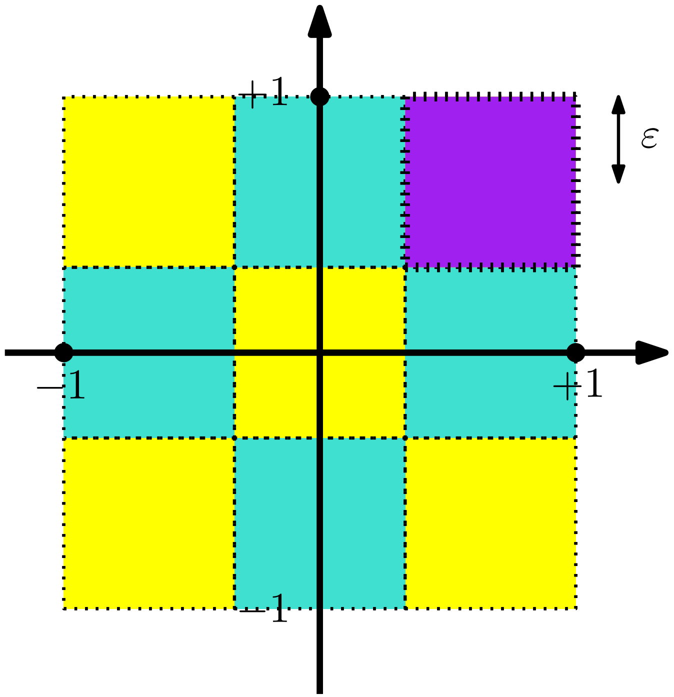
MRSS in expectation



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$$\mathbb{E}\left[\#\text{ subsets approximating any cube}\right] \geq 2^{n/2} \cdot \left(\frac{\varepsilon}{\sqrt{n/2}}\right)^d$$

$$= 2^{n/2 - d \log 1/\varepsilon - d/2 \log n/2} = 2^{O(n)} \text{ if } n \geq C d \log 1/\varepsilon$$

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< 1 if $n \leq cd\log 1/\varepsilon$ for c small enough

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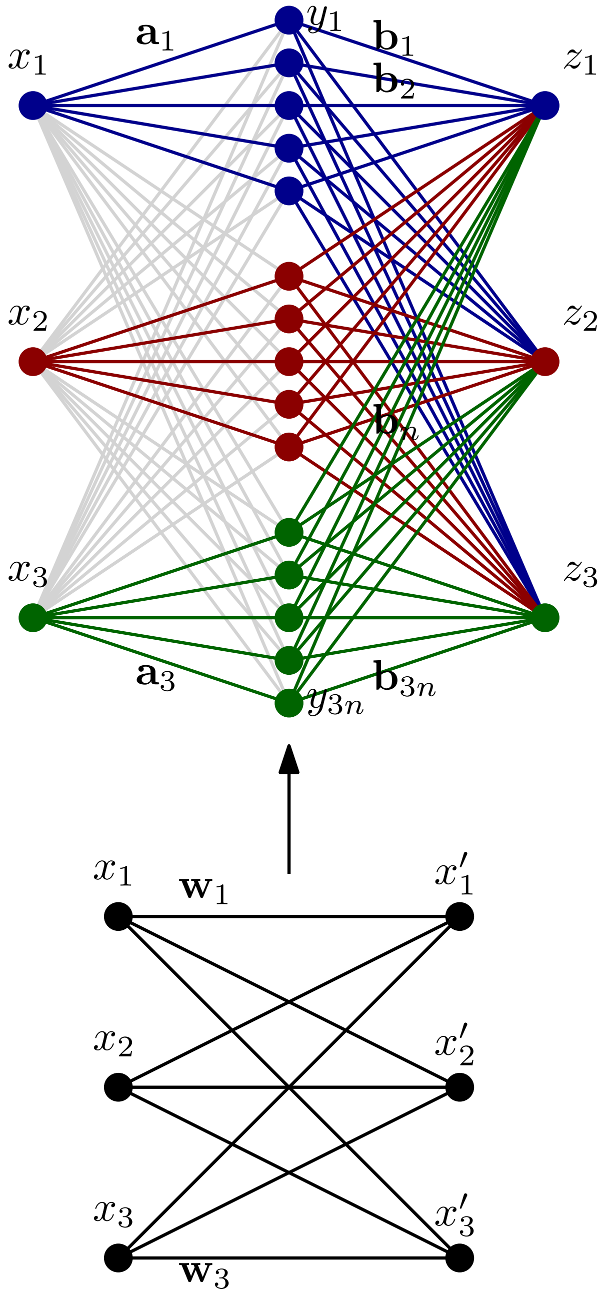
- What about approximating all the hypercube $[-1,1]^d$? The **union bound** is highly non-optimal

Apply MRSS for structured pruning



- $\mathbf{a}_i \sim \mathcal{N}(\mathbf{0}, I_n)$, $\mathbf{b}_i \sim \mathcal{N}(\mathbf{0}, I_d)$ (here, $d = 3$)

  (each neuron i has one $\mathbf{b}_i$)

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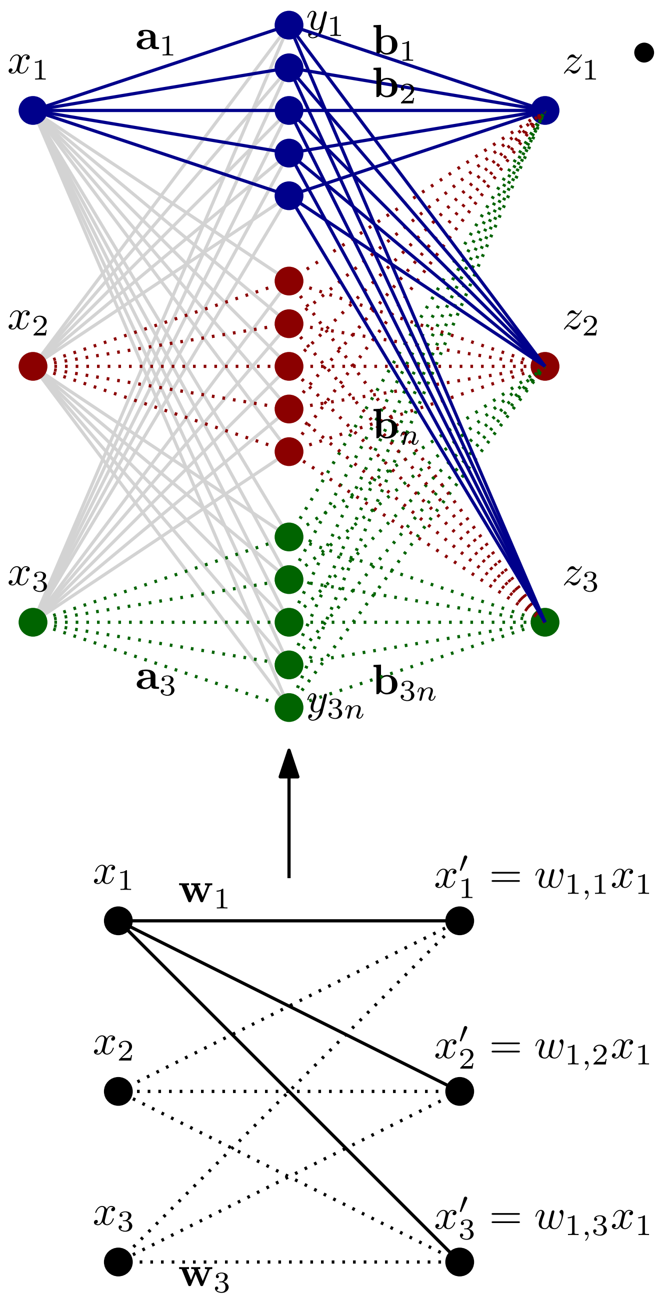

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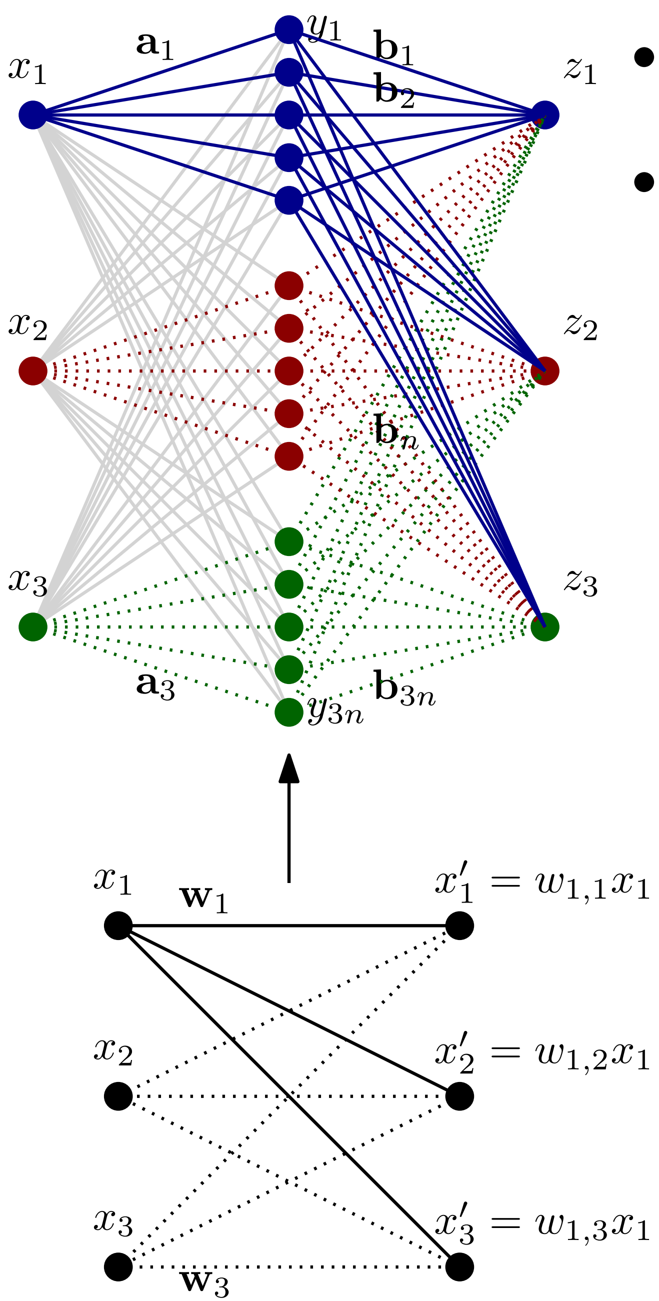

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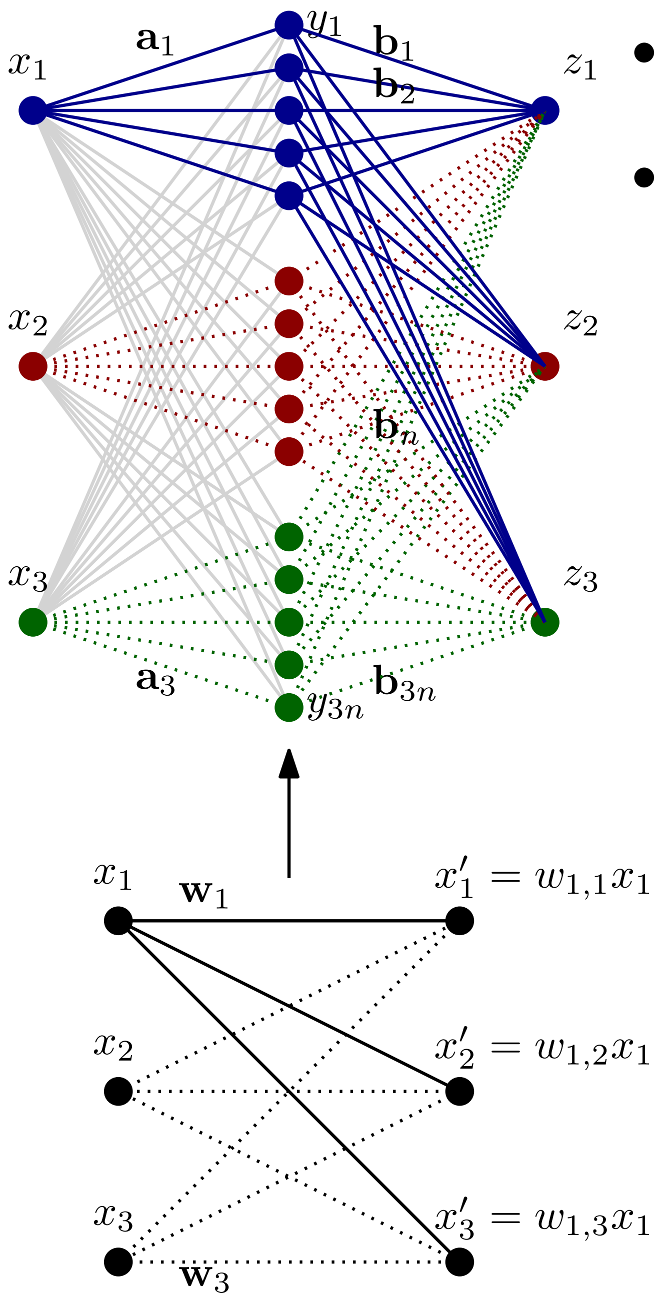

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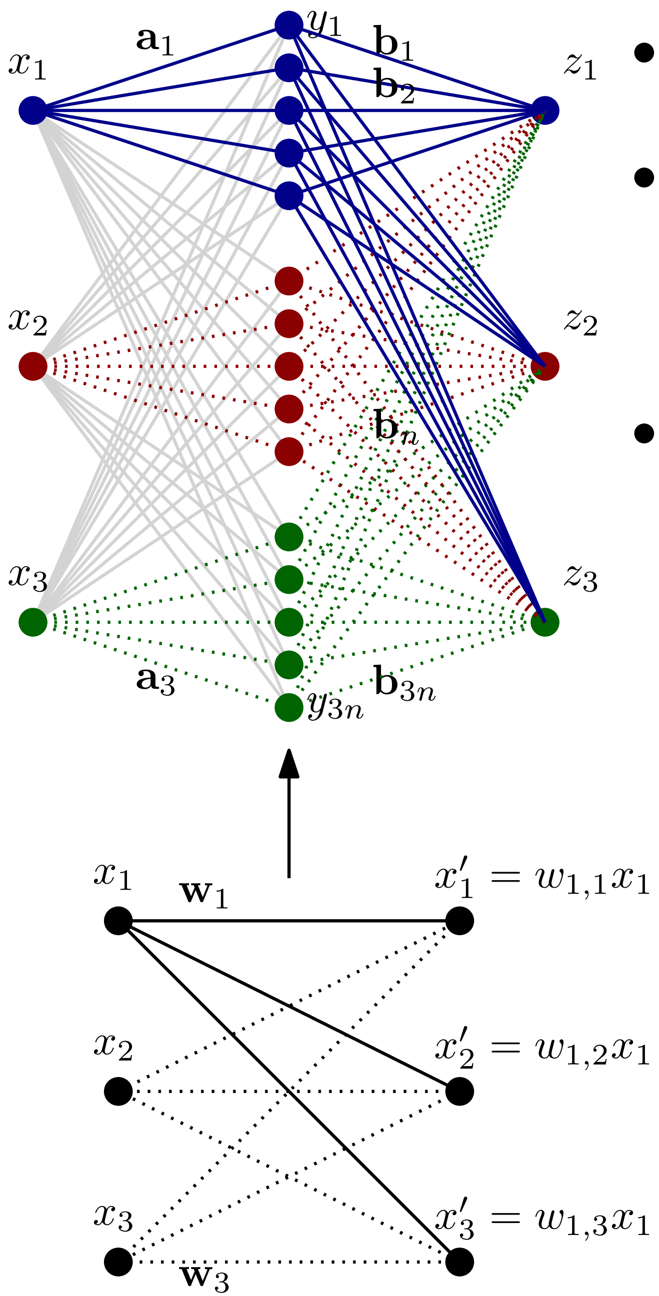

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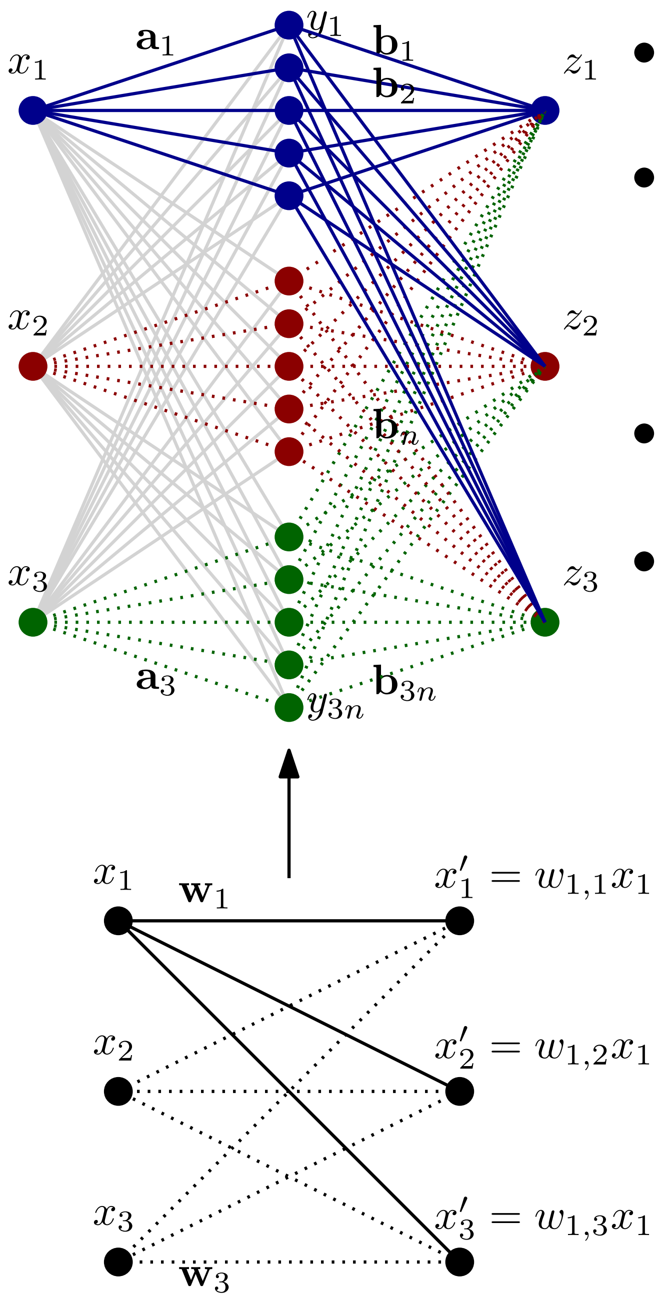

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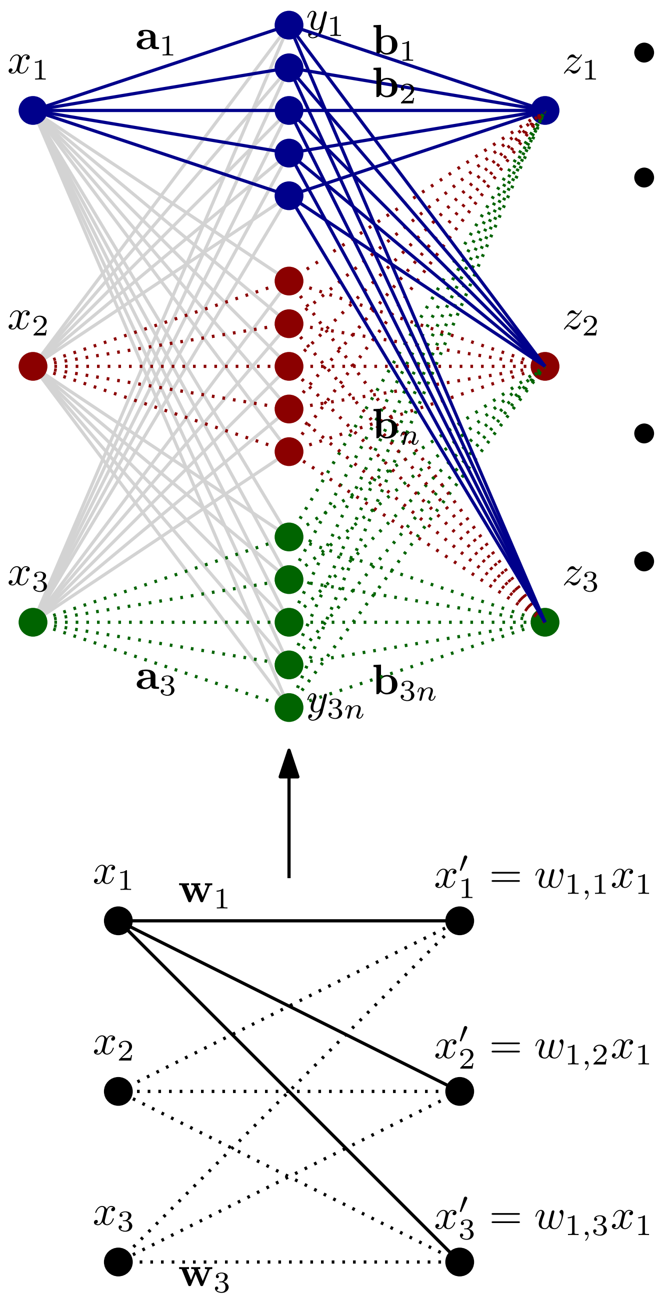

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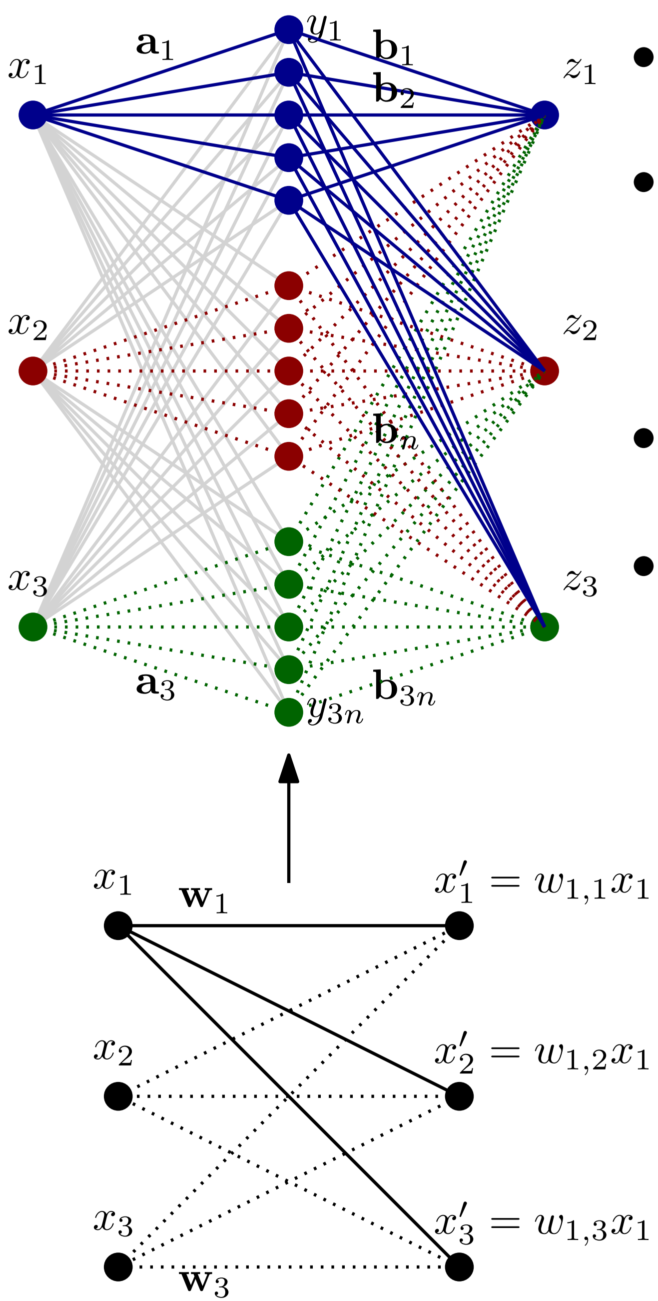

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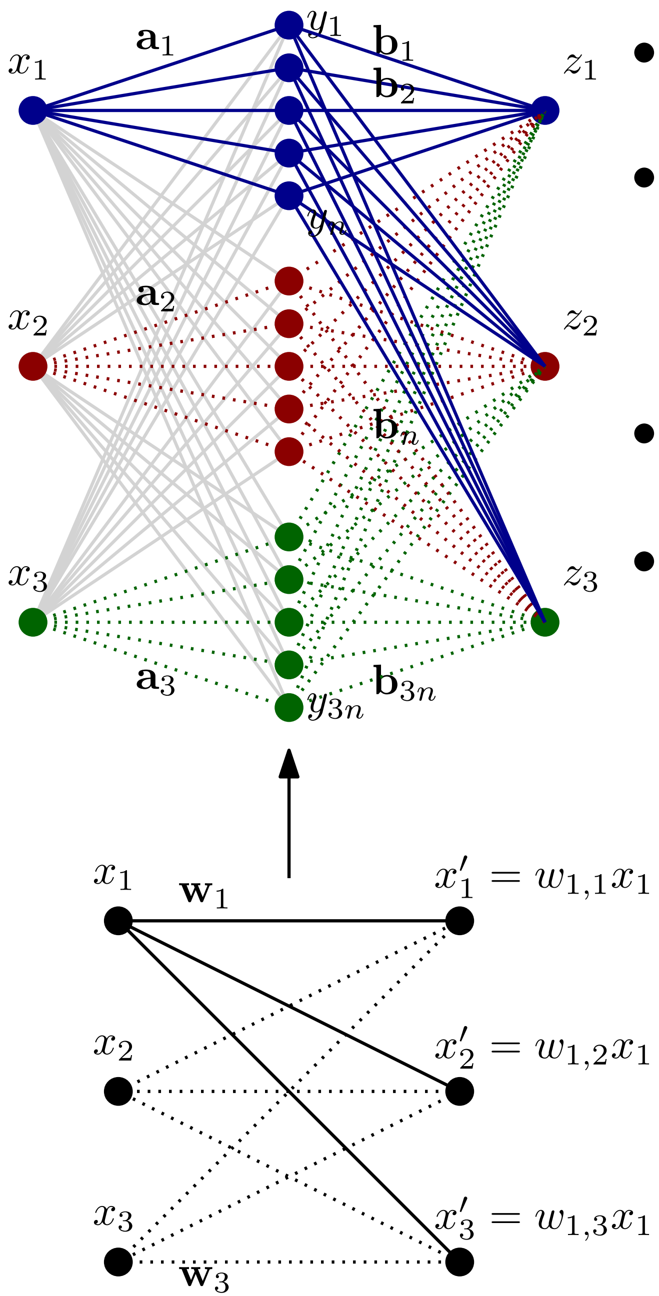

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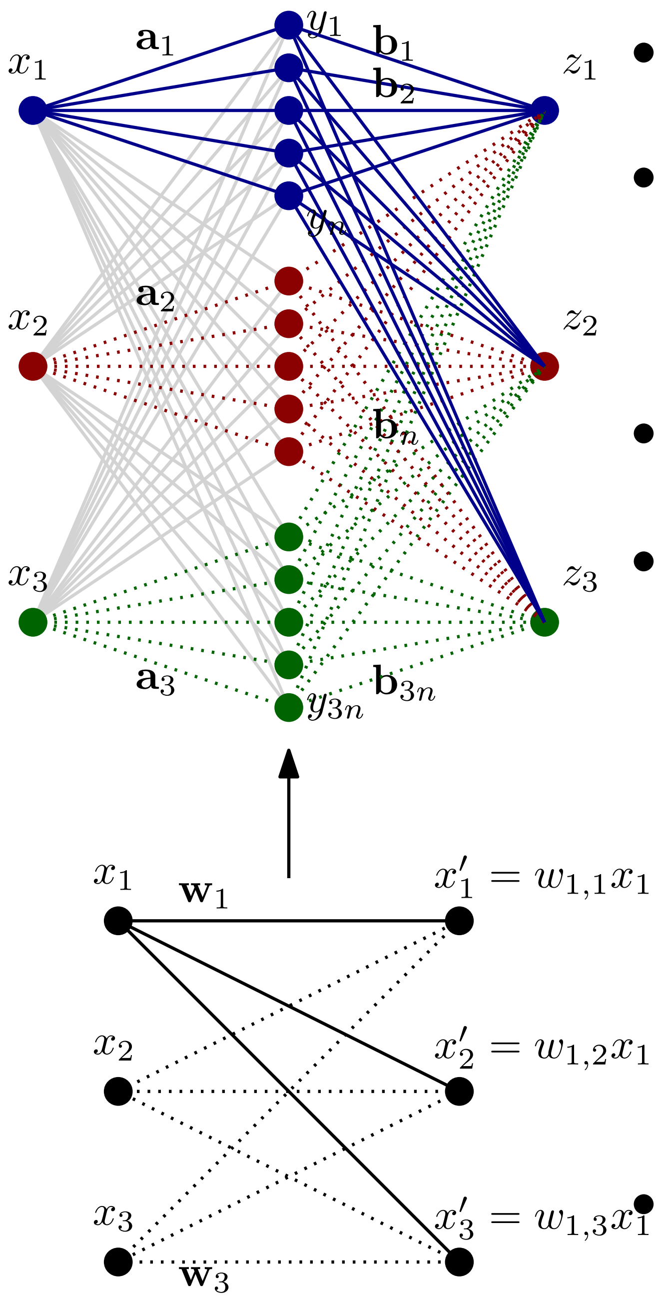

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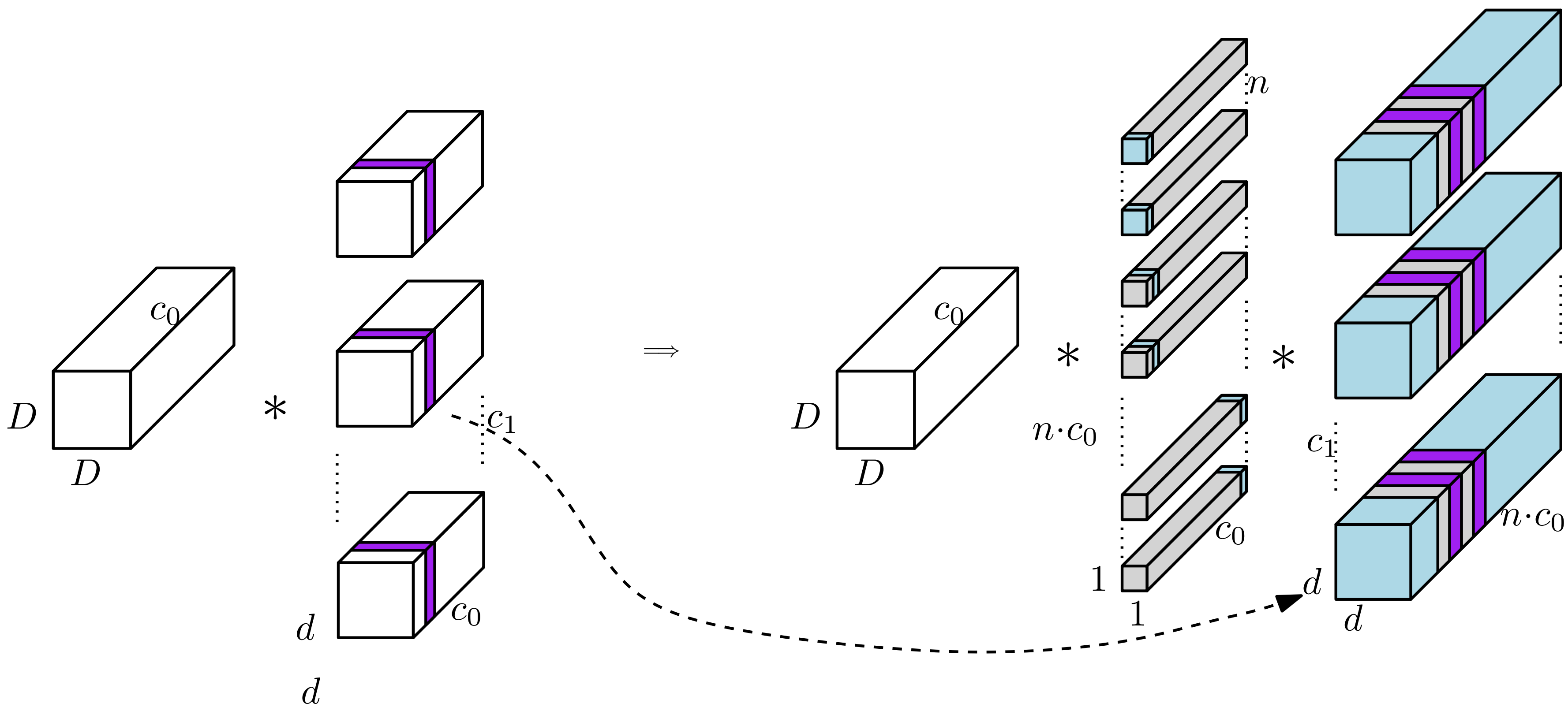
- **Result**: $n \geq \text{poly}(d) \cdot \text{polylog}(d\ell/\varepsilon)$

Convolutional neural networks (CNNs)

- **Generality**: There are even some results for CNNs. What other architectures can the SLTH be applied to?



Zhou et al. Algorithm

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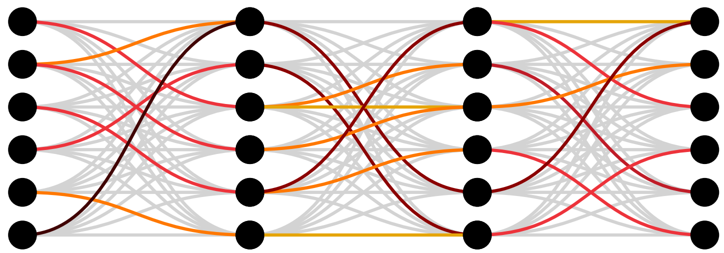

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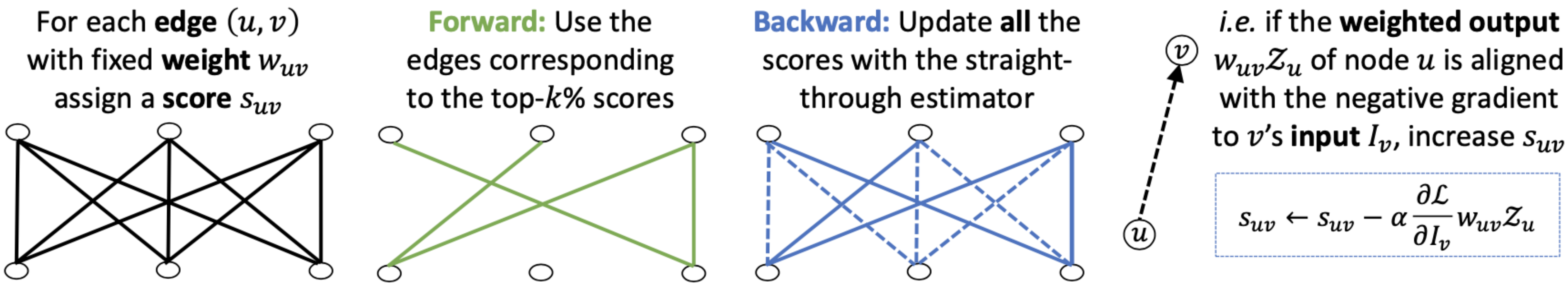
[Ramanujan et al 2020, CVPR]: Edge-popup Algorithm



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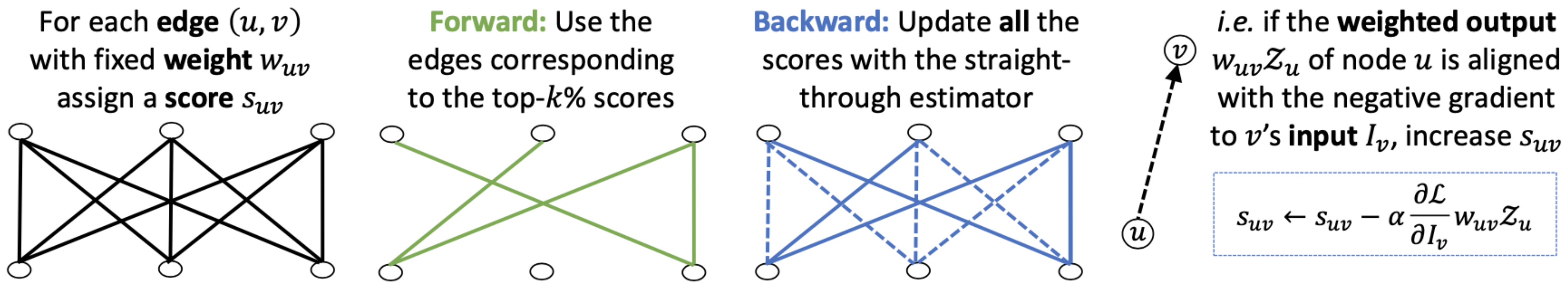

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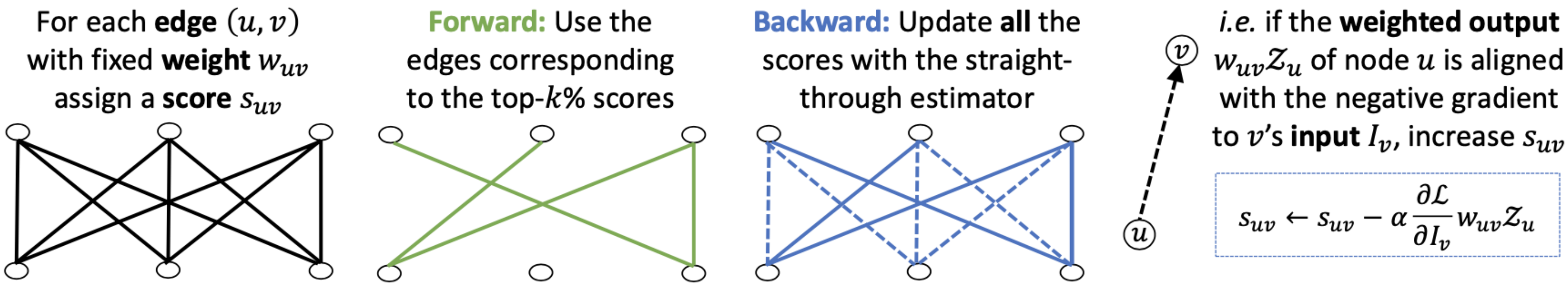

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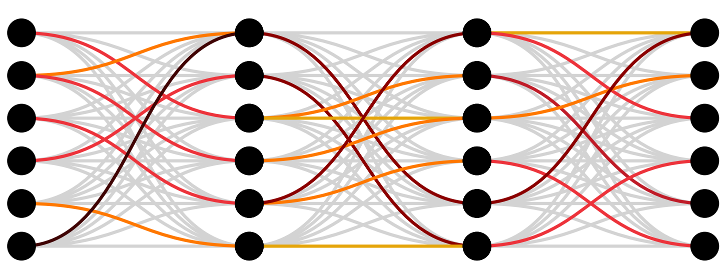
Crucially, their the final network only has the a size of $k\%$



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- **Open Problem**: None of the proofs have really moved beyond one layer. How much harder is it to approximate a deep neural network?

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I'm on sabbatical soon, if you want to work on this, let me know :)

Thank you!

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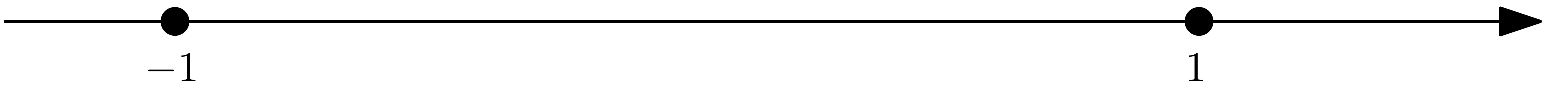

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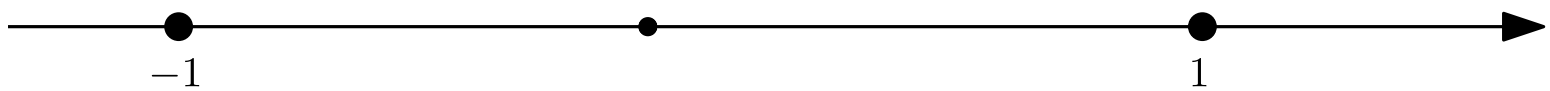

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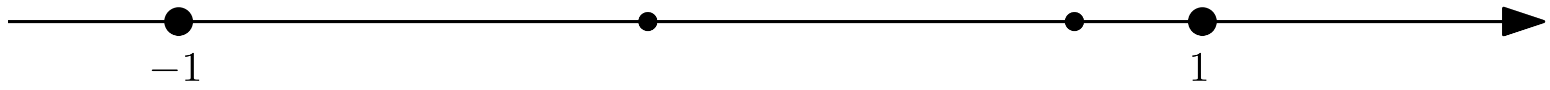

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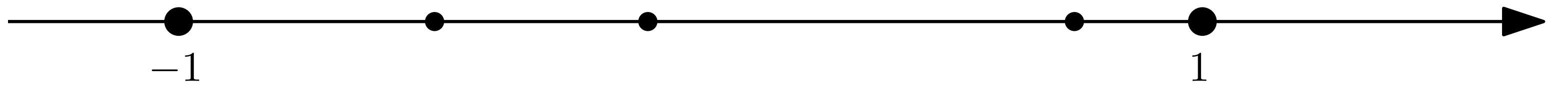

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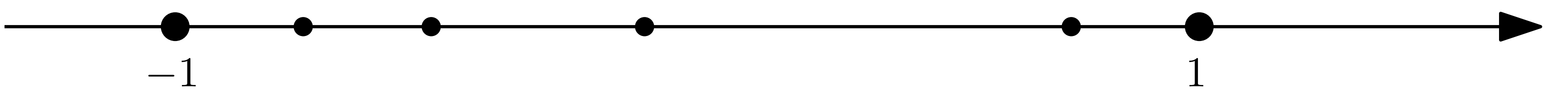

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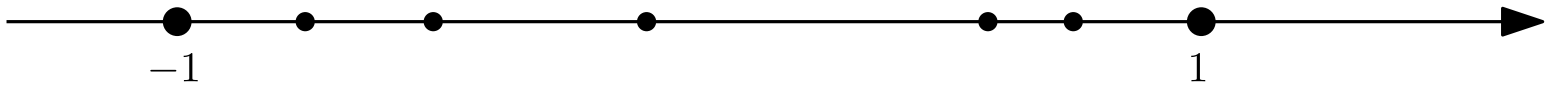

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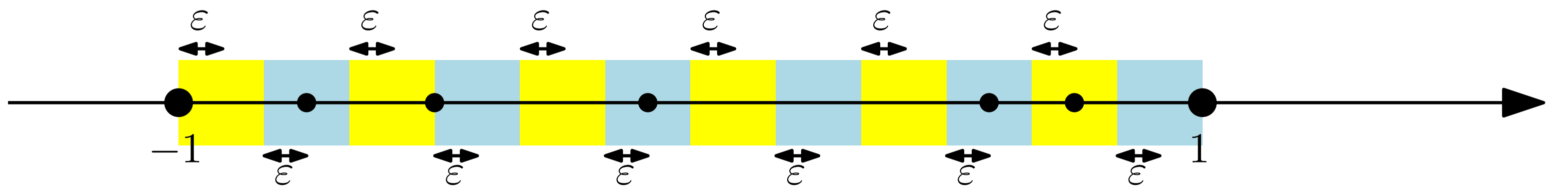

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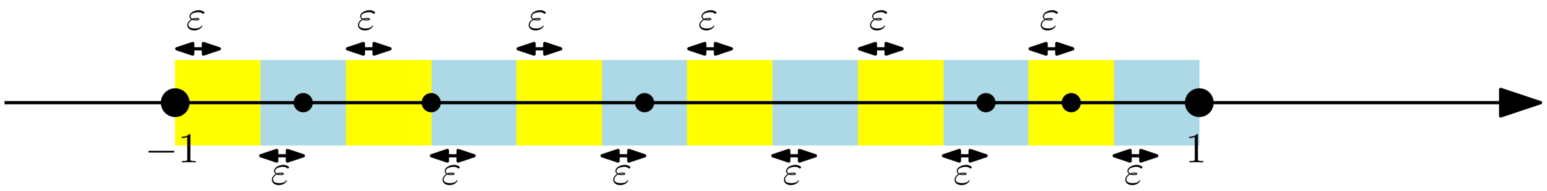

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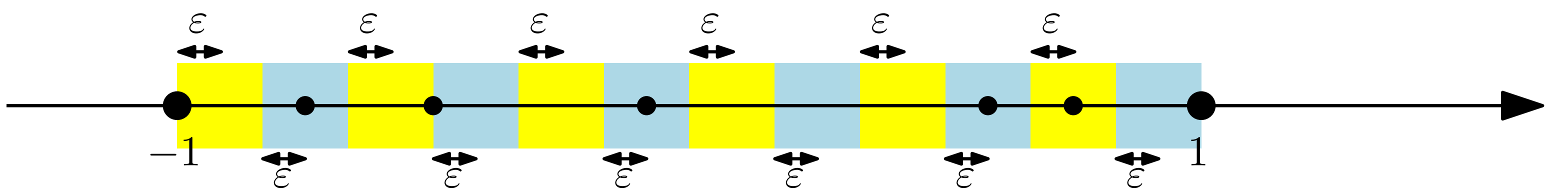

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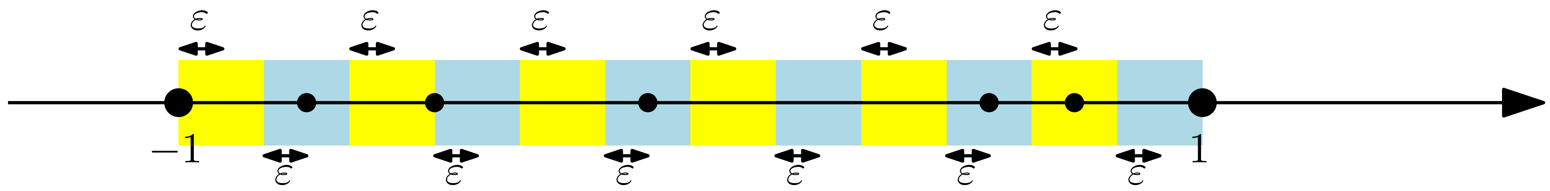

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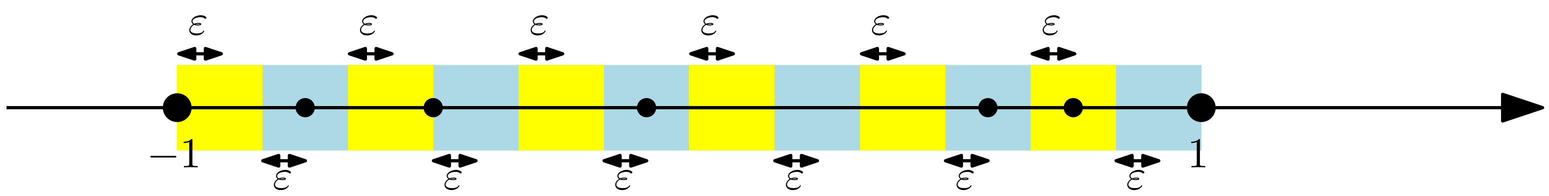

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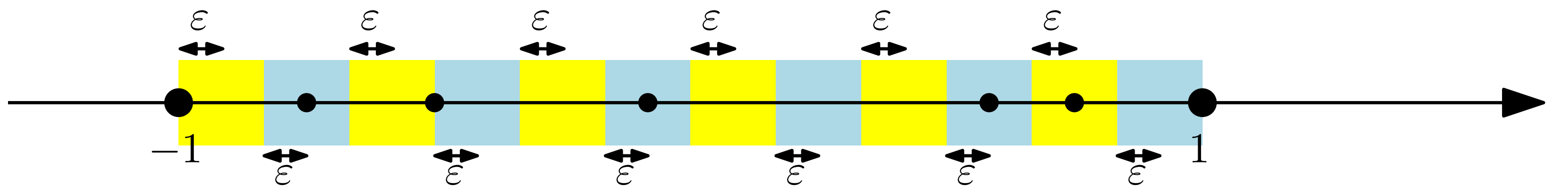

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For all $0 \leq t < n$, it holds that $\mathbb{E}\left[v_{t+1} \mid X_1, \ldots, X_t\right] \geq v_t \left[1 + \frac{1}{4}(1 - v_t)\right].$