On Continuation Methods for Non-Linear Multi-Objective Optimization

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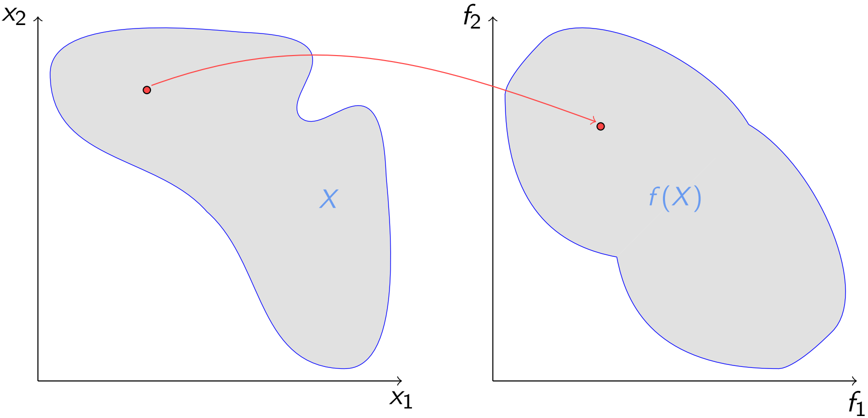
Non-Linear Multi-Objective Optimization



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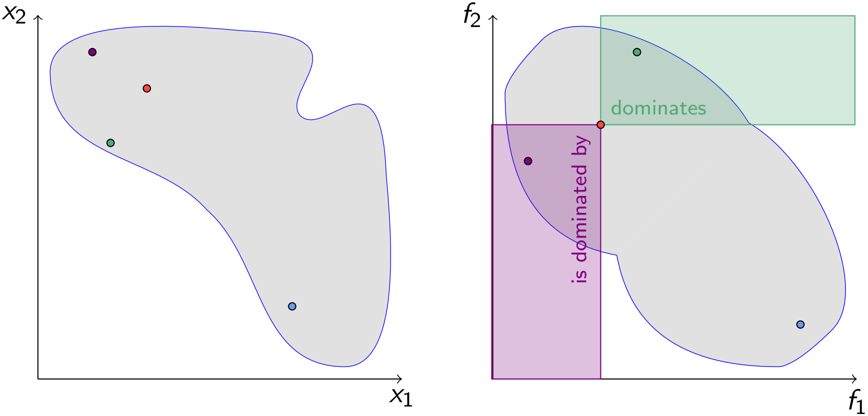
$$x \in X \subseteq \mathbb{R}^n$$

Non-Linear Multi-Objective Optimization



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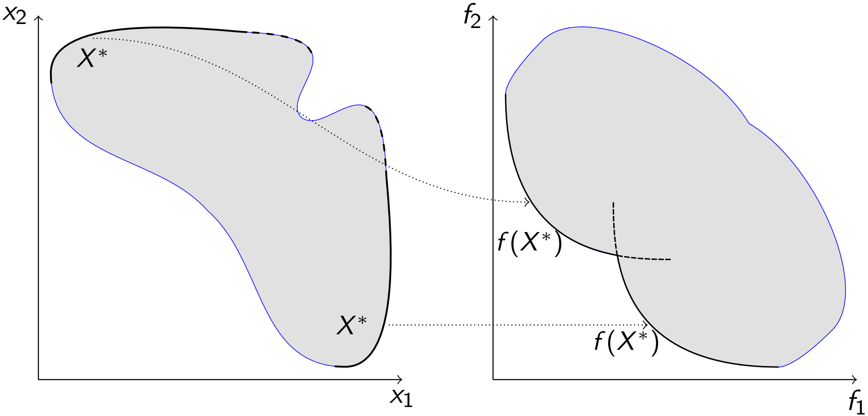

X^* set of non-dominated solutions: Pareto set (plain lines)
$f(X^*)$ set of non-dominated outcomes: Pareto front (plain lines)

Non-Linear Multi-Objective Optimization

General Non-Linear Multi-Objective Optimization (NLMOO) problem:

$$\begin{bmatrix} \min & f(x) \\ \text{s.t} & g(x) \leq 0 \\ & h(x) = 0 \\ & x \in \mathbb{R}^n \end{bmatrix} \tag{1}$$

Let $X = \{x \in \mathbb{R}^n | g(x) \leq 0, h(x) = 0\}$.

- Objective functions: $f : \mathbb{R}^n \to \mathbb{R}^k$,
- Inequality constraints: $g : \mathbb{R}^n \to \mathbb{R}^p$,
- Equality constraints: $h : \mathbb{R}^n \to \mathbb{R}^q$.

Functions may be non-linear.

What is continuation ?

Unformal definition

Local approximation/coverage of a manifold of solutions.

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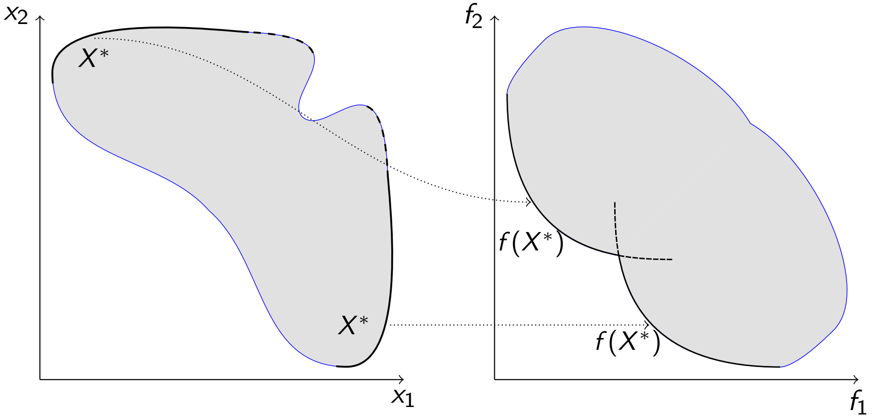
Unformal definition

Local approximation/coverage of a manifold of solutions.

- Local mean the use of local informations/observations,
- Solutions: of a system of equations, an optimization problem, . . . ; inducing (implicit) parameters,
- In NLMOO, when regular:
  - Two objectives $\rightarrow$ Manifold of dimension 1 (curves of solutions),
  - Three objectives $\rightarrow$ Manifold of dimension 2 (surfaces of solutions),
  - . . .

Continuation in Non-Linear Multi-Objective Optimization



X^* manifold of non-dominated solutions (plain lines)

$f(X^*)$ manifold of non-dominated outcomes (plain lines)

Scalarizing Methods

Non-Linear Multi-Objective Optimization problem

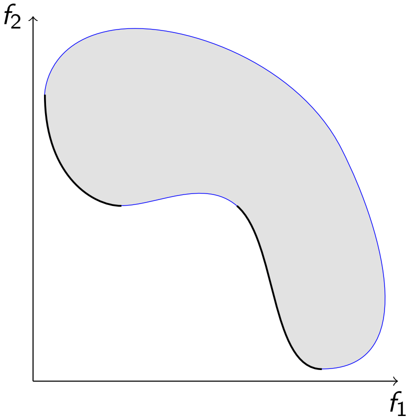
$$\begin{bmatrix} min & f(x) = (f_1(x), \dots, f_k(x)) \\ s.t & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

$$\Big\downarrow \text{Scalarizing}$$

$$\begin{bmatrix} min & \hat{f}(x, v) \\ s.t & \hat{g}(x, v) \leq 0 \\ & \hat{h}(x, v) = 0 \\ & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

Sequence of Mono-objective problems, $v \in \{v_1, v_2, \dots\}$

Scalarizing Methods: Examples



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  Minimize $\lambda f_1(x) + (1 - \lambda) f_2(x)$
- ϵ-Constraint:
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- Normal Boundary Intersection:
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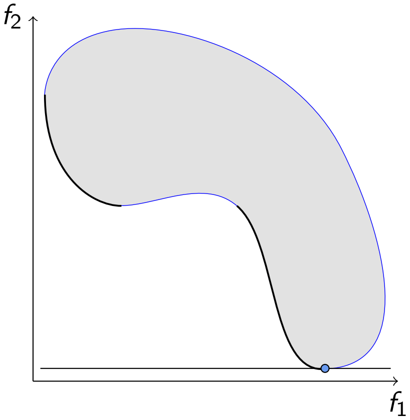

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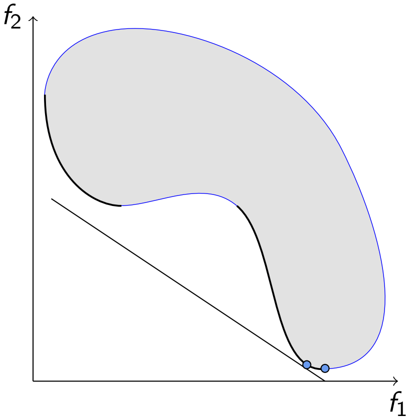

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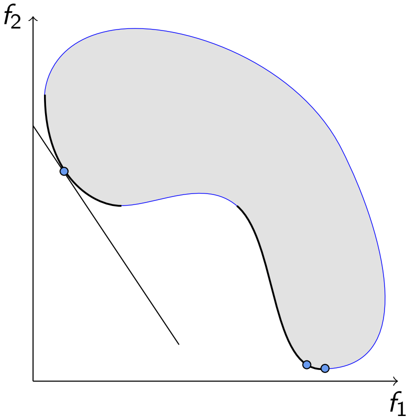

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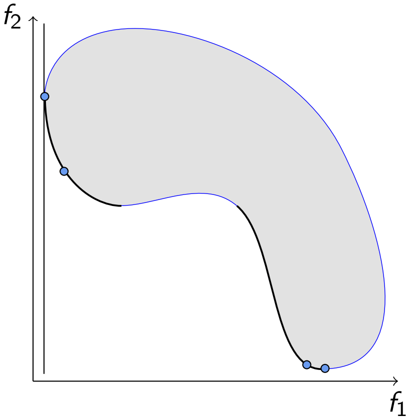

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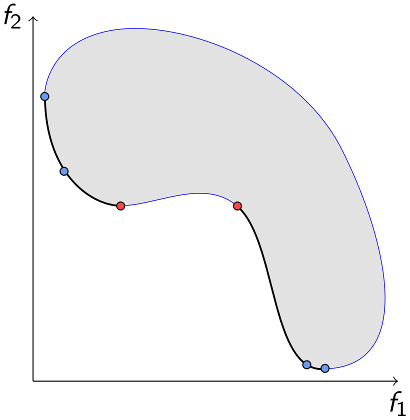

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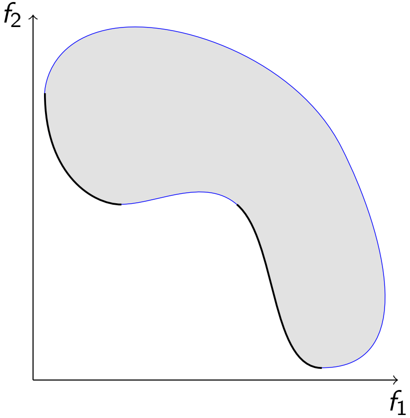

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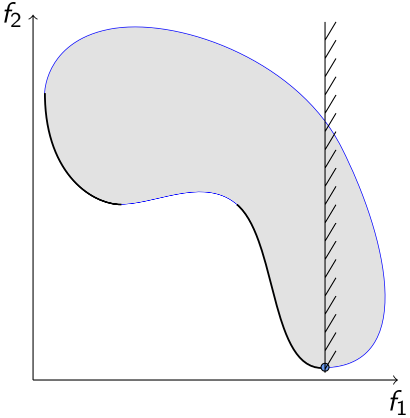

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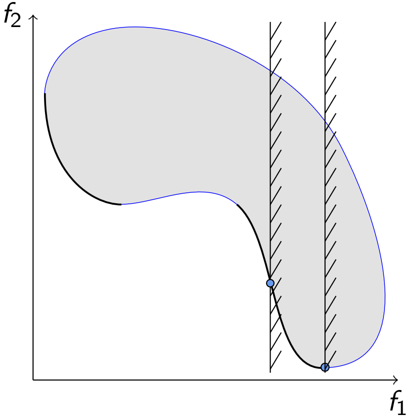

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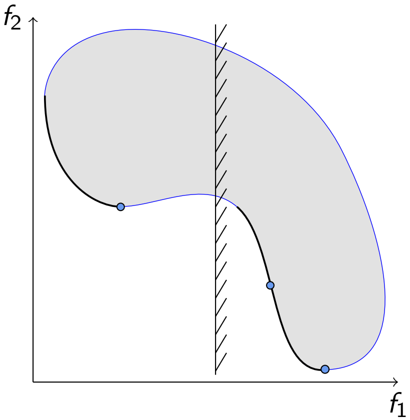

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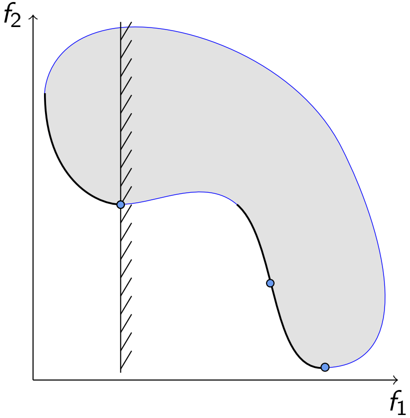

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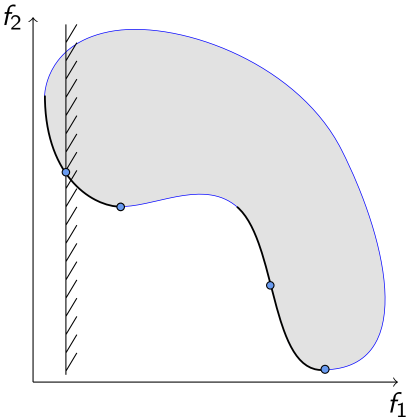

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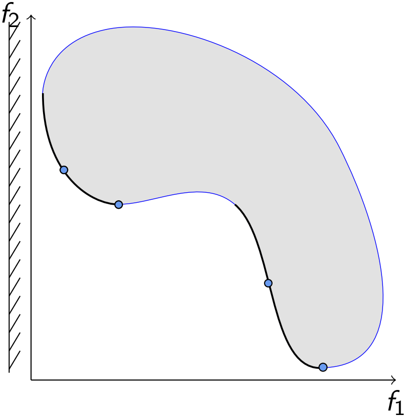

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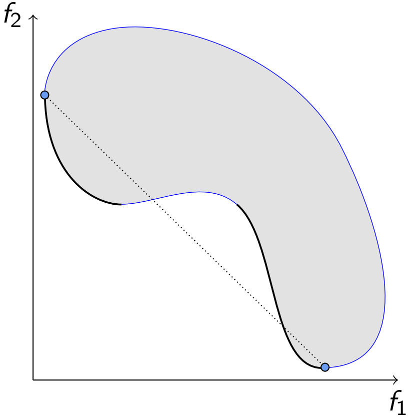

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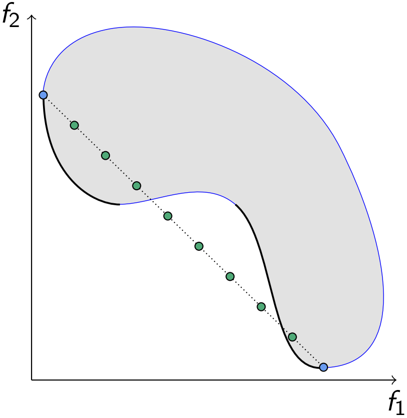

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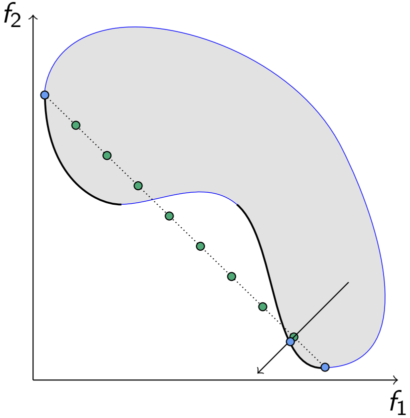

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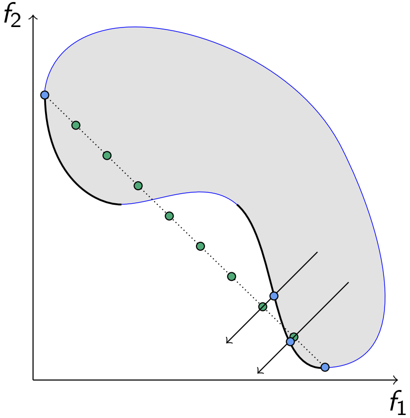

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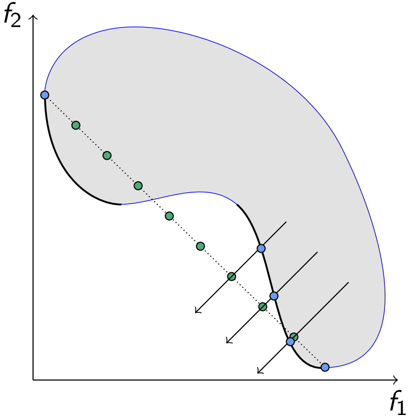

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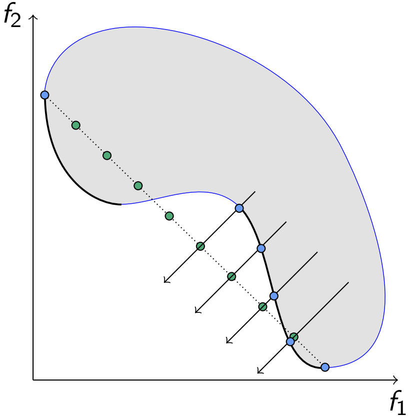

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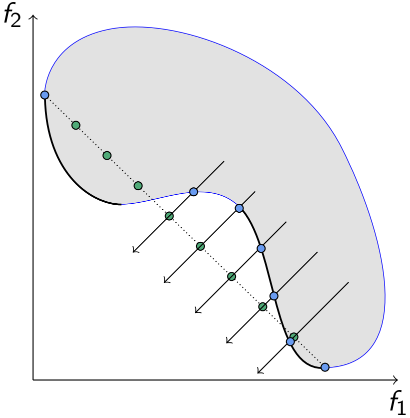

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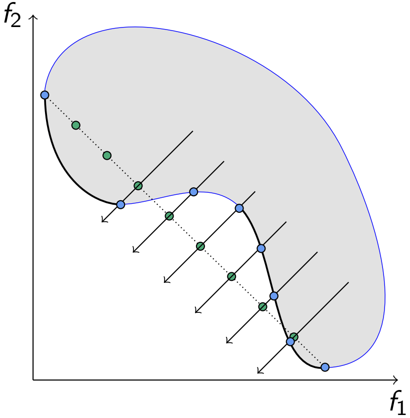

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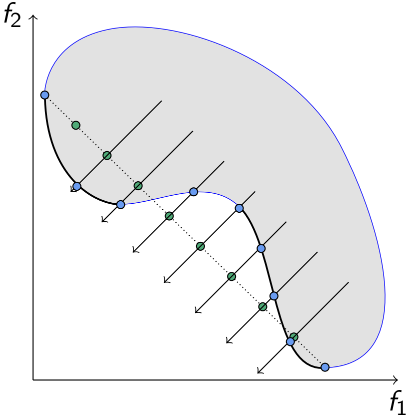
Scalarizing Methods: Examples



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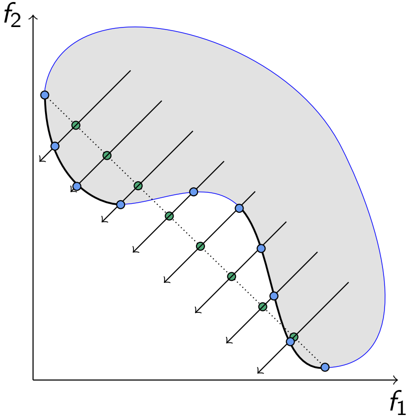

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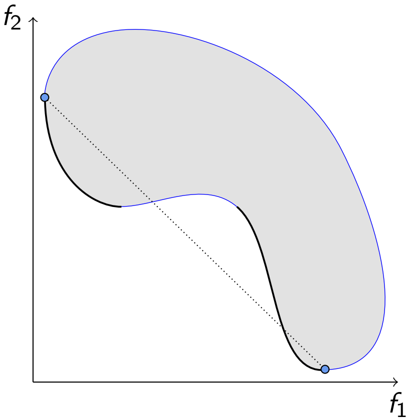

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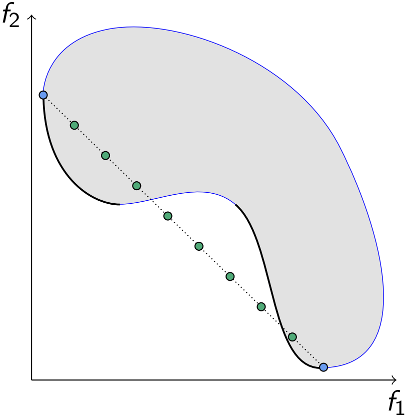

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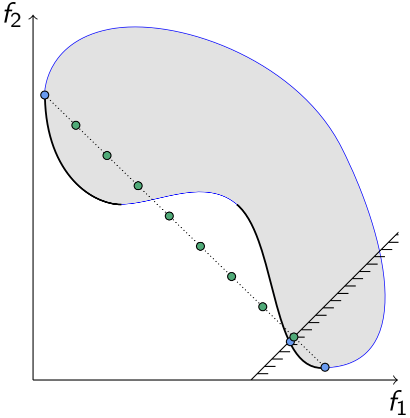

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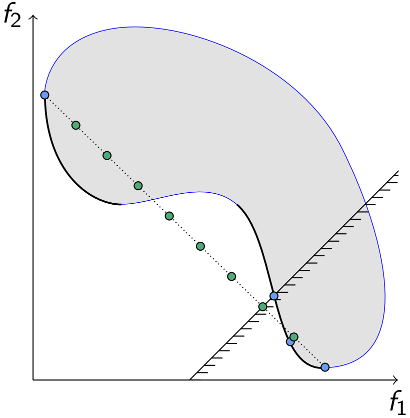

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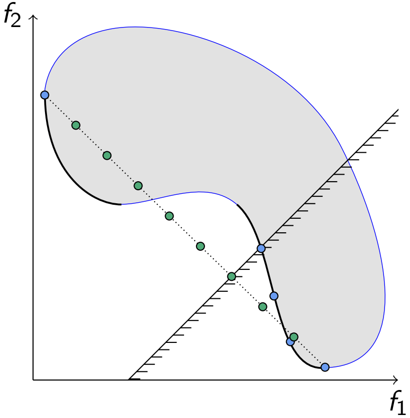

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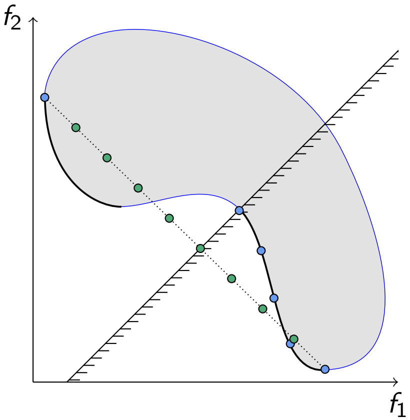

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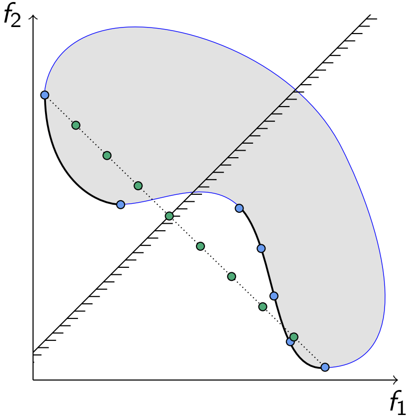

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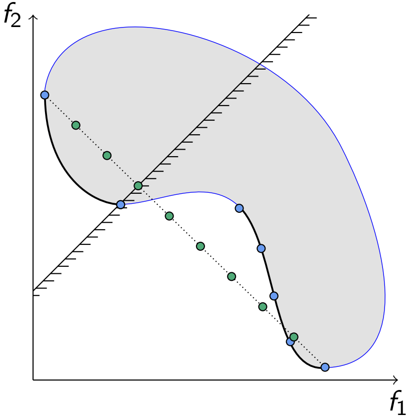

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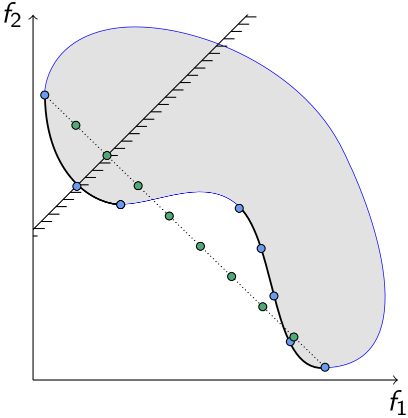

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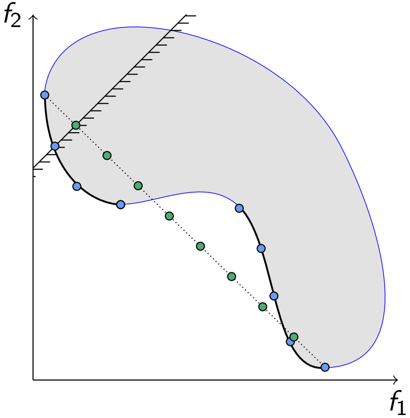

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Scalarizing Methods: Examples

$$\begin{bmatrix} min & f(x) \\ s.t & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

$$\Big\downarrow \text{Scalarizing}$$

$$\begin{bmatrix} min & \hat{f}(x, v) \\ s.t & \hat{g}(x, v) \leq 0 \\ & \hat{h}(x, v) = 0 \\ & g(x) \leq 0 \\ & h(x) = 0 \end{bmatrix}$$

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Parametric Optimization

Parametric Optimization problem:

$$\left[\begin{array}{ll} \min & f(x,v) \\ \text{s.c} & h(x,v) = 0 \\ & g(x,v) \leq 0 \\ & x \in \mathbb{R}^n \end{array}\right]$$

$f : \mathbb{R}^{n+m} \to \mathbb{R}$ and $v \in V \subseteq \mathbb{R}^m$ vector of parameters.
Continuation is natural in such applications:

- Parameters are explicit.
- Use of local informations interesting.
- Optimal solutions are usually computed as solutions to first order optimality conditions.

Parametric Optimization

First order conditions:

$$\begin{aligned}
\nabla_x f(x,v)\lambda + \nabla_x g(x,v)r + \nabla_x h(x,v)s &= 0\\
(\forall i = 1,\dots,p)\ g_i(x,v)r_i &= 0\\
(\forall i = 1,\dots,q)\ h_i(x,v) &= 0\\
\lambda^T\lambda + r^T r + s^T s - 1 &= 0
\end{aligned}$$

With $x \in X \subseteq \mathbb{R}^n$, $v \in \mathbb{R}^m$, $\lambda \in \mathbb{R}_+$, $r \in \mathbb{R}_+^p$ and $s \in \mathbb{R}^q$.
System of $n+m+1+p+q$ variables with $n+p+q+1$ equations:

$$F(x) = 0,\ F : \mathbb{R}^{\delta+m} \to \mathbb{R}^{\delta}$$

Regular $\to$ m-dimensional manifold of solutions

State of the art

Literature on Parametric Optimization:

- Algorithms [Rao and Papalambros, 1989, Rakowska et al., 1991].
- Singularity detections [Lundberg and Poore, 1993].
- Multi-Parametric [Domínguez et al., 2010].

Towards Multi-Objective Optimization:

- Tackling Multi(Bi)-Objective optimization [Rakowska et al., 1993].

State of the art

First order optimality conditions:
Parametric problem based on Weighted Sum

$$\begin{aligned}
\nabla_x f_1(x)\lambda_1 + \nabla_x f_2(x)\lambda_2 + \nabla_x g(x) r + \nabla_x h(x) s &= 0\\
(\forall i = 1, \dots, p)\ g_i(x) r_i &= 0\\
(\forall i = 1, \dots, q)\ h_i(x) &= 0\\
\lambda^T\lambda + r^T r + s^T s - 1 &= 0
\end{aligned}$$

NLMOO first order conditions

$$\begin{aligned}
\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla g(x) r + \nabla h(x) s &= 0\\
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\end{aligned}$$

Continuation methods and applications

Continuation methods used to solve underconstrained systems of equations.

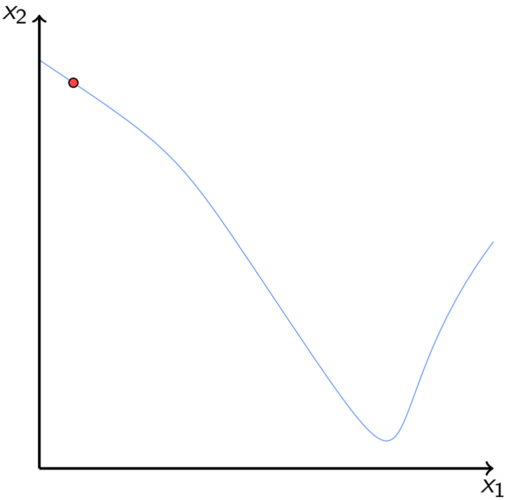
General problem

$$F(x) = 0, \; F : \mathbb{R}^{n+m} \to \mathbb{R}^n$$

Solutions form a *m*-dimensional manifold.
Appears in:

- Study of parameters in differential equations;
- Homotopy for solving polynomial systems;
- Non-Linear Optimization (interior-point methods, **parametric optimization**);
- ...

Continuation methods example: Predictor/Corrector



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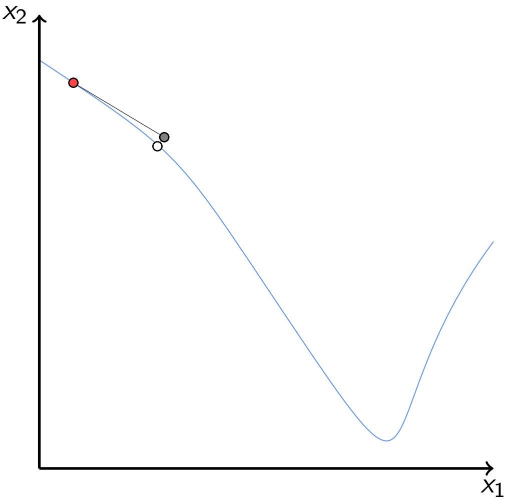

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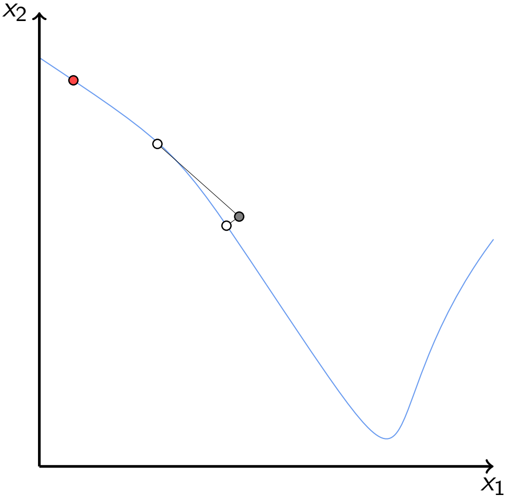

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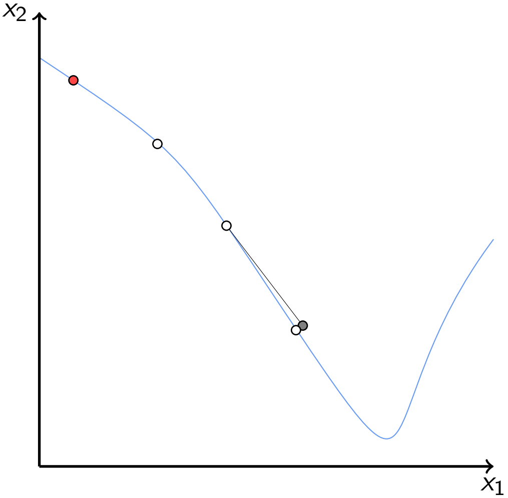

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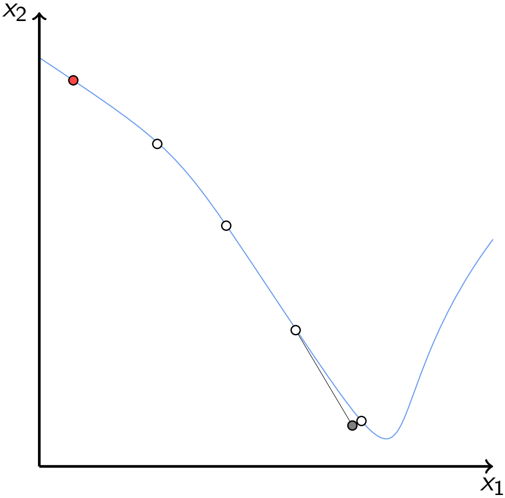

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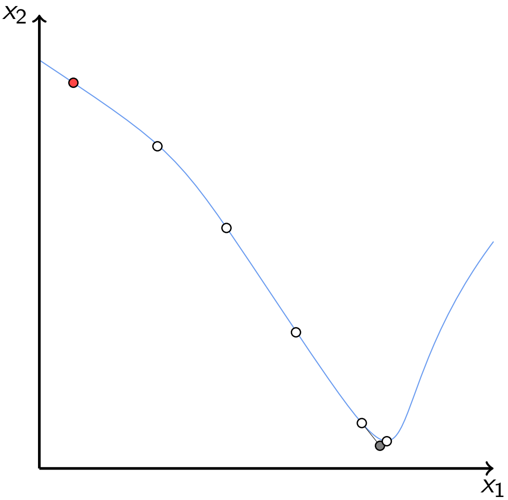

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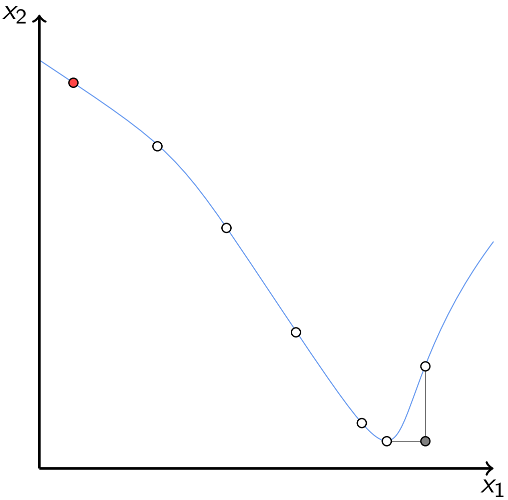

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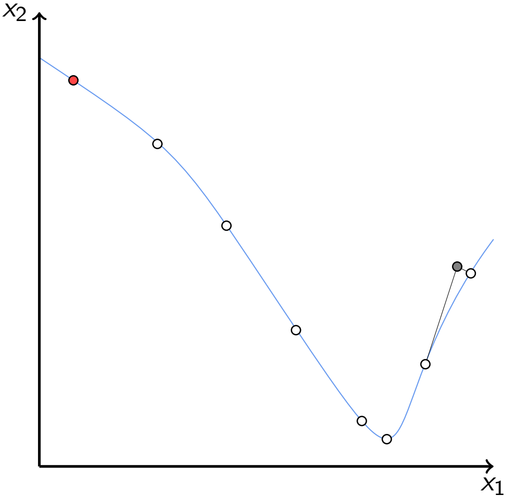


NLMOO hybridized with Continuation Methods

Continuation methods:

- For first order conditions of NLMOO [Hillermeier, 2001].

Applications in Metaheuristics:

- Curve-based Genetic Algorithm [Harada et al., 2007]
- PSO and continuation [Schütze et al., 2008]
- Steepest Descent (HCS) as continuation [Schütze et al., 2009]

Applications in Global methods:

- Recovering algorithm [Schütze et al., 2005]
- Bi-objective method inspired by NBI [Pereyra, 2009, Pereyra et al., 2013]
- Global Search [Lovison, 2011, Lovison, 2012]

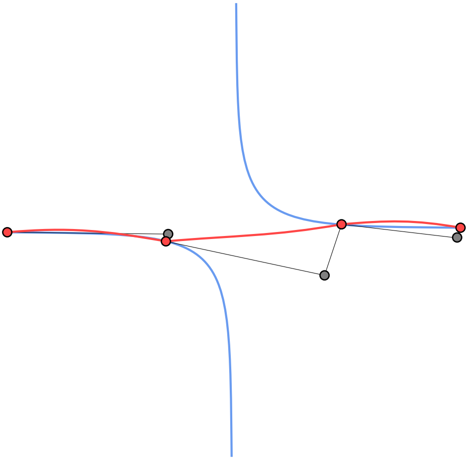
Summary

- Continuation methods + NLMOO promising,
- Help for both metaheuristics and global algorithms,

Summary

- ⊕ Continuation methods + NLMOO promising,
- ⊕ Help for both metaheuristics and global algorithms,
- ⊖ Few actually consider inequality constraints,
- ⊖ Few gives certification of the continuity.
  - False representation of the manifold.
  - Loss of solutions.

Summary



Summary

- ⊕ Continuation methods + NLMOO promising,
- ⊕ Help for both metaheuristics and global algorithms,
- ⊖ Few actually consider inequality constraints,
- ⊖ Few gives certification of the continuity.

Certification can be (numerically) achieved:

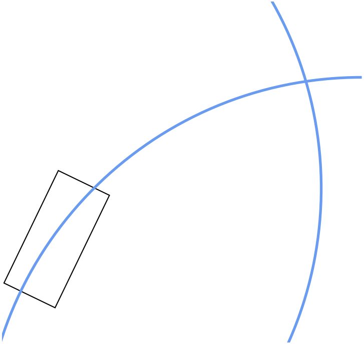
- Smale α-theory or Kantorovich theorem $\rightarrow$ maximal step [Beltrán and Leykin, 2012, Faudot and Michelucci, 2007],
- Interval Analysis and parametric Interval Newton operators [Kearfott and Xing, 1994, Martin et al., 2013].

Goal

Towards a certified and rigorous continuation method for (inequality) constrained NLMOO.

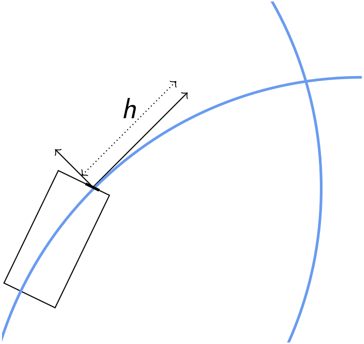
Here, restricted to the Bi-Objective case.

ParCont: Certified Continuation with Parallelotopes



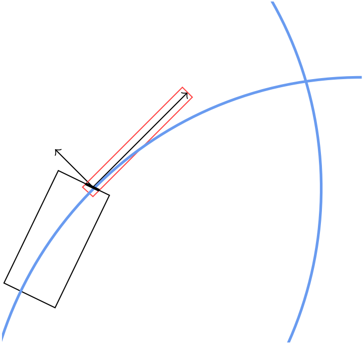
- Parallelotopes and parametric Interval Newton [Goldsztejn and Granvilliers, 2010]
  - Used in Constraint Programming,
  - Based on interval analysis,
  - Spouse the shape of the manifold.
- ParCont: Parallelotope-based Continuation [Martin et al., 2013]
- Singularities

ParCont: Certified Continuation with Parallelotopes



- Parallelotopes and parametric Interval Newton [Goldsztejn and Granvilliers, 2010]
- ParCont: Parallelotope-based Continuation [Martin et al., 2013]
  - Equivalent PC method,
  - Builds locally new parallelotopes along the manifold,
  - Connects two consecutive parallelotopes.
- Singularities

ParCont: Certified Continuation with Parallelotopes



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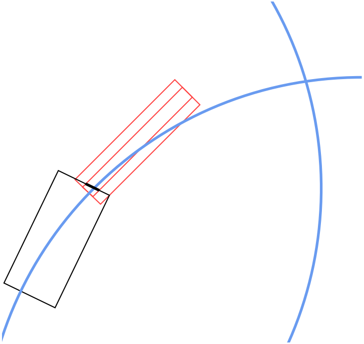

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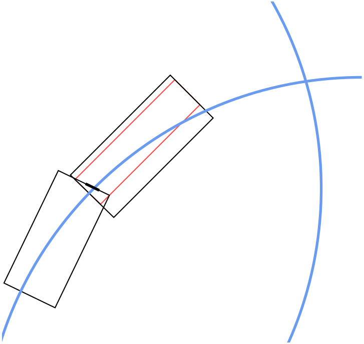

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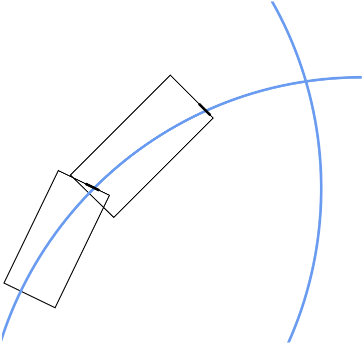
  - Builds locally new parallelotopes along the manifold,
  - Connects two consecutive parallelotopes.
- Singularities

ParCont: Certified Continuation with Parallelotopes



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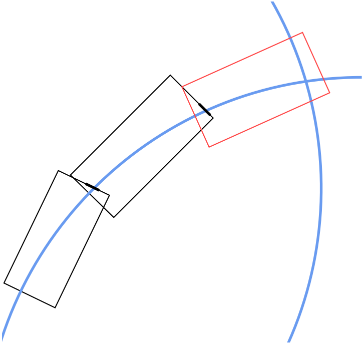

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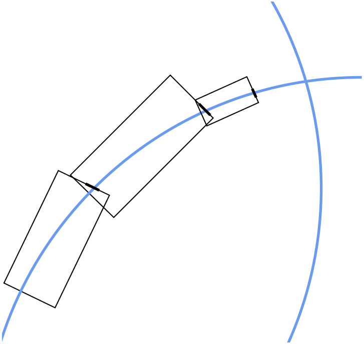

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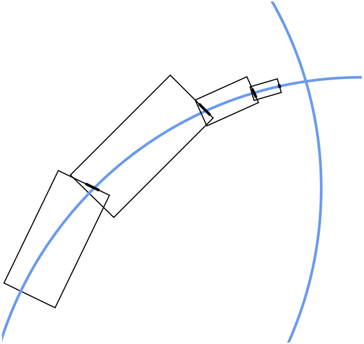

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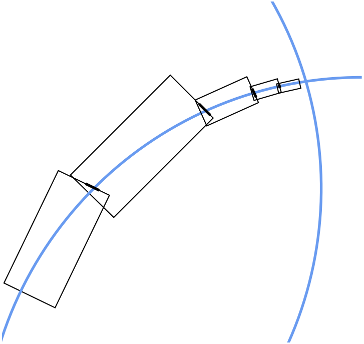

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- ParCont: Parallelotope-based Continuation [Martin et al., 2013]
- Singularities
  - Can not certify singularities.

Optimality conditions

Let the system of first order optimality conditions:

$$\begin{aligned}
\nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla g(x) r + \nabla h(x) s &= 0 \\
(\forall i = 1, \dots, p)\ g_i(x) r_i &= 0 \\
(\forall i = 1, \dots, q)\ h_i(x) &= 0 \\
\lambda^T\lambda + r^T r + s^T s - 1 &= 0
\end{aligned}$$

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Singularity when there exists i with $r_i = 0$ and $g_i(x) = 0$: change in the set of active constraints.

Problem

ParCont can not handle inequality constraints.

Towards a certified active set management strategy

Dealing with inequalities [Rakowska et al., 1993]

Definition

Let $\bar{\mathcal{A}} \subseteq \{1, 2, \dots, p\}$ be the set of active constraints at a feasible solution x. Let $\bar{g}$ and $\bar{r}$ be the induced inequality vector and weights.

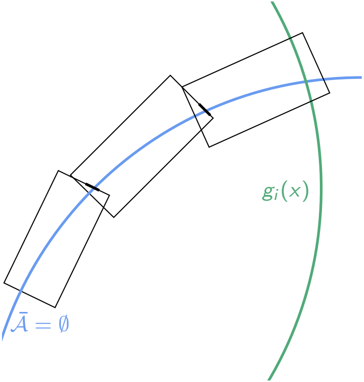
To deal with singularities from change in the active constraint set:

- Solve the system:

$$\begin{aligned} \nabla f_1(x)\lambda_1 + \nabla f_2(x)\lambda_2 + \nabla \bar{g}(x)\bar{r} + \nabla h(x)s &= 0 \\ (\forall i \in \bar{\mathcal{A}})\ g_i(x) &= 0 \\ (\forall i = 1, \dots, q)\ h(x) &= 0 \\ \lambda^T\lambda + r^Tr + s^Ts - 1 &= 0 \end{aligned} \tag{2}$$

- Change the set $\bar{\mathcal{A}}$ when activating/disactivating a constraint.

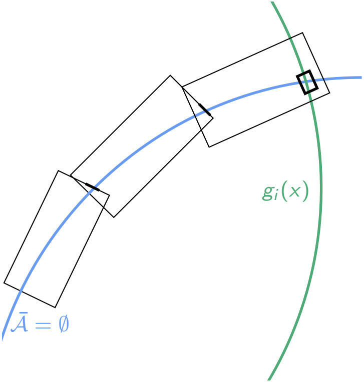
Detecting change in the active set



Example:

- Detection of a possible activation ($g_i(x) = 0$),
- Certify the activation: Interval Newton,
- Change $\bar{\mathcal{A}}$, isolate the activation, orient the continuation ($r_i > 0$),
- Restart the Continuation.

Detecting change in the active set



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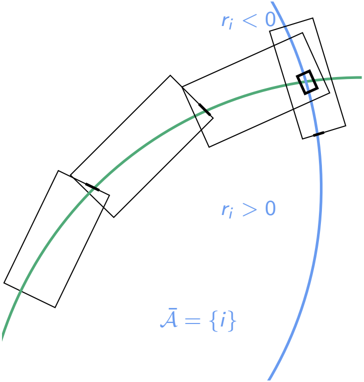

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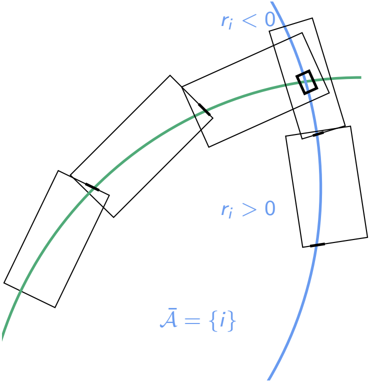

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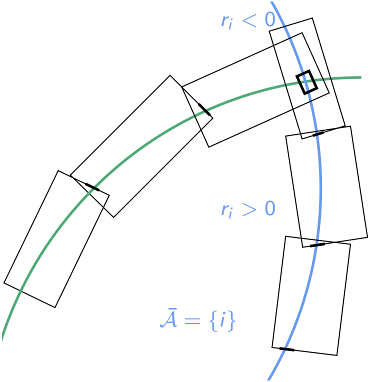

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- Change $\bar{\mathcal{A}}$, isolate the activation, orient the continuation ($r_i > 0$),
- Restart the Continuation.

Implementation

ParCont is implemented in C++, with:

- RealPaver API [Granvilliers and Benhamou, 2006],
- Gaol interval arithmetic library [Goualard, 2006],
- Lapack linear algebra library [Anderson et al., 1999],

Towards a method for recovering a (local) Pareto optimal set starting from one single solution.

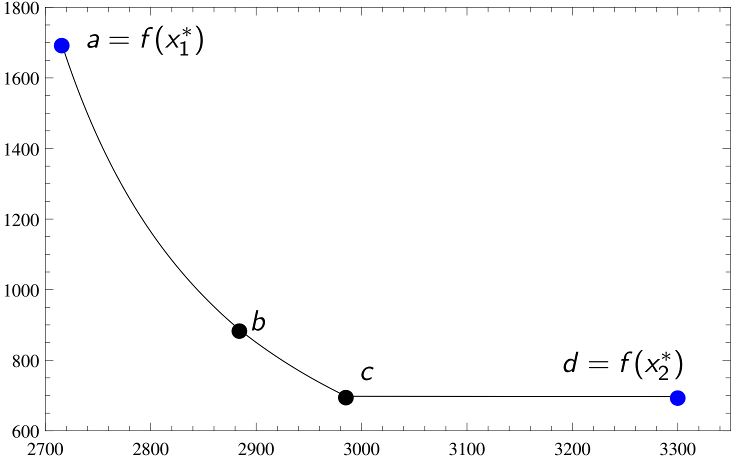
Illustrative problem

The following problem corresponds to the design of a speed reducer:

$$\left[\begin{array}{l} \min\ f_1(x) = 0.7854x_1x_2^2(\frac{10x_3^2}{3} + 14.933x_3 - 43.0934) \\ \qquad - 1.508x_1(x_6^2 + x_7^2) + 7.477(x_6^3 + x_7^3) + 0.7854(x_4x_6^2 + x_5x_7^2) \\ \min\ f_2(x) = \sqrt{(745x_4/x_2x_3)^2 + 1.69 \times 10^7}/0.1x_6^3 \\ \text{s.t}\ g_1(x) = \frac{1}{x_1x_2^2x_3} - \frac{1}{27} \leq 0;\ g_2(x) = \frac{1}{x_1x_2^2x_3^2} - \frac{1}{397.5} \leq 0 \\ g_3(x) = \frac{x_4^3}{x_2x_3x_6^4} - \frac{1}{1.93} \leq 0;\ g_4(x) = \frac{x_5^3}{x_2x_3x_7^4} - \frac{1}{1.93} \leq 0 \\ g_5(x) = x_2x_3 - 40 \leq 0;\ g_6(x) = \frac{x_1}{x_2} - 12 \leq 0 \\ g_7(x) = 5 - \frac{x_1}{x_2} \leq 0;\ g_8(x) = 1.9 - x_4 + 1.5x_6 \leq 0 \\ g_9(x) = 1.9 - x_5 + 1.1x_7 \leq 0;\ g_{10}(x) = f_1(x) - 3300 \leq 0 \\ g_{11}(x) = \sqrt{(745x_5/x_2x_3)^2 + 1.575 \times 10^8}/0.1x_7^3 - 1100 \leq 0 \end{array}\right],$$

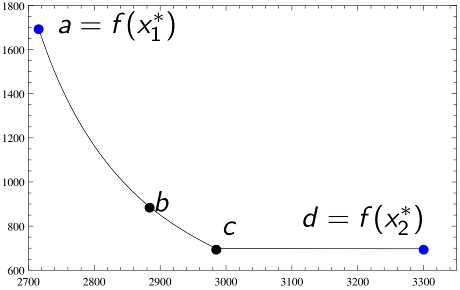
with $2.6 \leq x_1 \leq 3.6$, $0.7 \leq x_2 \leq 0.8$, $17 \leq x_3 \leq 28$, $7.3 \leq x_4, x_5 \leq 8.3$, $2.9 \leq x_6 \leq 3.9$ and $5 \leq x_7 \leq 5.5$. Variable domains are considered as inequality constraints.

Illustrative problem: results



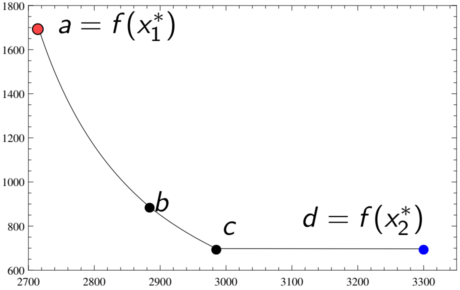
0.25 seconds and 48 parallelotopes

Illustrative problem: results



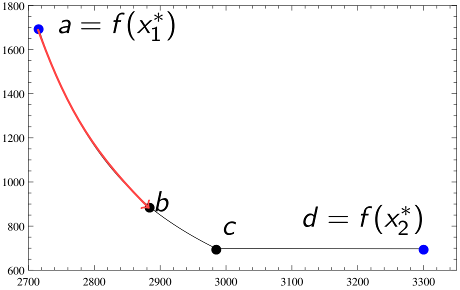
Constraint	$a \to a$	$a \to b$	$b \to b$	$b \to c$	$c \to c$	$c \to d$	$d \to d$
g_8	$-$	$-$	$+$	$+$	$+$	$+$	$+$
$\underline{g}_{x_4}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\underline{g}_{x_6}$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\overline{g}_{x_6}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$\underline{g}_{x_3}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
g_{10}	$-$	$-$	$-$	$-$	$-$	$-$	$+$

Illustrative problem: results



Constraint	$a \to a$	$a \to b$	$b \to b$	$b \to c$	$c \to c$	$c \to d$	$d \to d$
g_8	$-$	$-$	$+$	$+$	$+$	$+$	$+$
$\underline{g}_{x_4}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\underline{g}_{x_6}$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\overline{g}_{x_6}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$\underline{g}_{x_3}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
g_{10}	$-$	$-$	$-$	$-$	$-$	$-$	$+$

Illustrative problem: results



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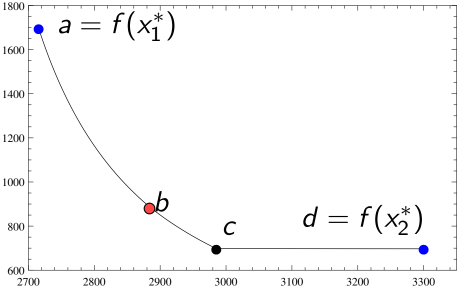
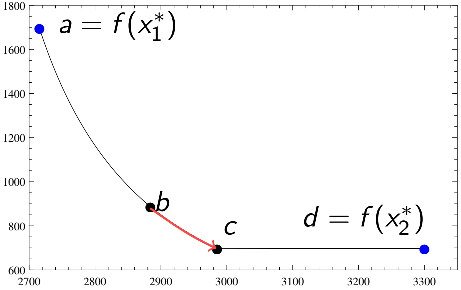
$\underline{g}_{x_4}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\underline{g}_{x_6}$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\overline{g}_{x_6}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$\underline{g}_{x_3}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
g_{10}	$-$	$-$	$-$	$-$	$-$	$-$	$+$

Illustrative problem: results



Constraint	$a \to a$	$a \to b$	$b \to b$	$b \to c$	$c \to c$	$c \to d$	$d \to d$
g_8	−	−	+	+	+	+	+
$\underline{g}_{x_4}$	+	+	+	−	−	−	−
$\underline{g}_{x_6}$	+	−	−	−	−	−	−
$\overline{g}_{x_6}$	−	−	−	−	+	+	+
$\underline{g}_{x_3}$	+	+	+	+	+	−	−
g_{10}	−	−	−	−	−	−	+

Illustrative problem: results



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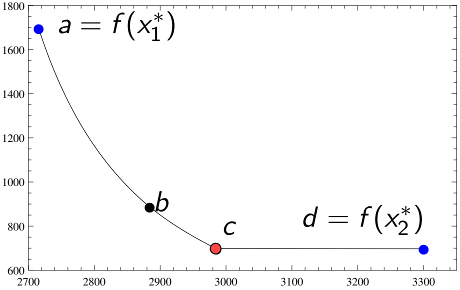
$\underline{g}_{x_6}$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\overline{g}_{x_6}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$\underline{g}_{x_3}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
g_{10}	$-$	$-$	$-$	$-$	$-$	$-$	$+$

Illustrative problem: results



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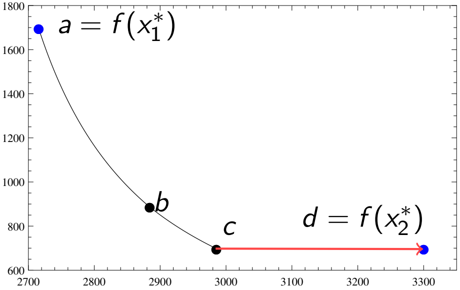
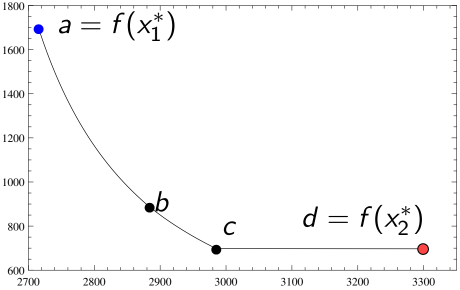


Constraint	$a \to a$	$a \to b$	$b \to b$	$b \to c$	$c \to c$	$c \to d$	$d \to d$
g_8	$-$	$-$	$+$	$+$	$+$	$+$	$+$
$\underline{g}_{x_4}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\underline{g}_{x_6}$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\overline{g}_{x_6}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$\underline{g}_{x_3}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
g_{10}	$-$	$-$	$-$	$-$	$-$	$-$	$+$

Illustrative problem: results



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$\underline{g}_{x_4}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\underline{g}_{x_6}$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$\overline{g}_{x_6}$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
$\underline{g}_{x_3}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
g_{10}	$-$	$-$	$-$	$-$	$-$	$-$	$+$

Summary

Informations:

- Each step of ParCont has $O(n^3)$ time complexity,
- Compared to non-certified methods, it has to use a smaller step length.

Pros and Cons:

- $\oplus$ ParCont ables to produce certified enclosures of Pareto-optimal solutions,
- $\oplus$ Local optimality is proven for each enclosure,
- $\oplus$ Use only local informations.
- $\ominus$ Required twice continuously differentiable objectives and constraints,
- $\ominus$ Some singularities not handled,
- $\ominus$ Limited to 1-dimensional manifolds (bi-objective problems).

Summary

We have seen that:

- Many state of art approaches attempt to parameterize Pareto-Optimal solutions,
- Continuation methods and NLMOO promising in different applications,
- A certified continuation method ParCont for bi-objective problems, dealing with change in active set of constraints, is proposed.

Next ?

- Integration of ParCont in a global method: only one point per connected components is required,

Summary

We have seen that:

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Next ?

- Integration of ParCont in a global method: only one point per connected components is required,
- Adaptation of ParCont to 3-Objectives.

On Continuation Methods for Non-Linear Multi-Objective Optimization

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Laurent GRANVILLIERS and Christophe JERMANN
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Nantes, 26 June 2013

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Ecole des Mines de Nantes

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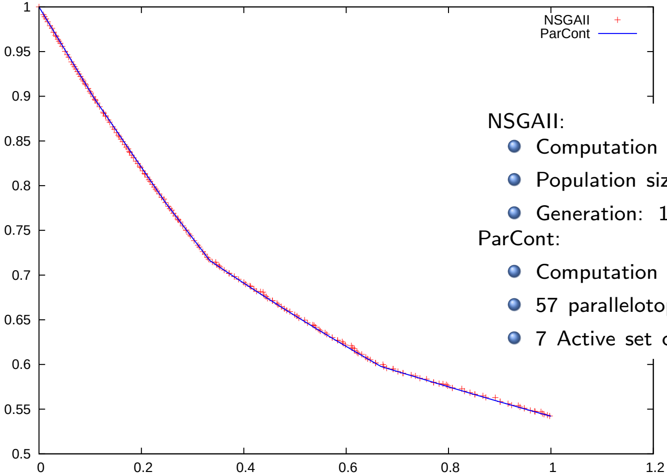
Experiments

CTP1 [Deb et al., 2001]: Standard bi-objective problem with 2 variables.

$$\left[\begin{array}{ll} \text{min} & f_1(x) = x_1 \\ \text{min} & f_2(x) = (1 + x_2)\exp(-x_1/(1 + x_2)) \\ \text{s.t} & g_1(x) = 1 - f_2(x)/(0.858\exp(-0.541 f_1(x))) \leq 0 \\ & g_2(x) = 1 - f_2(x)/(0.728\exp(-0.295 f_1(x))) \leq 0 \\ & x_1, x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}\right]$$

Start ParCont at f_2^*.

Experiments



NSGAII:

- Computation time: 0.25 s,
- Population size: 200,
- Generation: 100.

ParCont:

- Computation time: 0.0925 s,
- 57 parallelotopes,
- 7 Active set changes.

Experiments

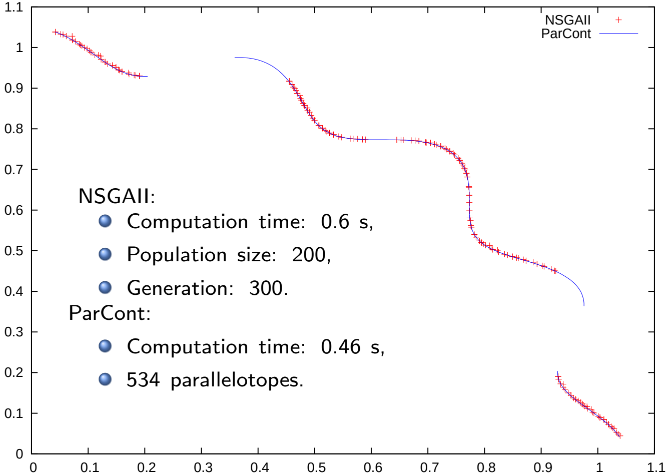
Tanaka [Tanaka et al., 1995]: Bi-objective problem with 2 variables.

$$\left[\begin{array}{ll} \min & f_1(x) = x_1 \\ \min & f_2(x) = x_2 \\ \text{s.t} & g_1(x) = -x_1^2 - x_2^2 + 1 + 0.1\cos(16\arctan(x_1/x_2)) \leq 0 \\ & g_2(x) = 2x_1^2 + 2x_2^2 - 1 \leq 0 \\ & x_1, x_2 \leq \pi \\ & x_1, x_2 \geq 0 \end{array}\right]$$

Disconnected Pareto-front. ParCont started at:

$$x^A = \begin{pmatrix} 0.042 \\ 1.038 \end{pmatrix},\ x^B = \begin{pmatrix} 0.586 \\ 0.774 \end{pmatrix},\ x^C = \begin{pmatrix} 1.039 \\ 0.043 \end{pmatrix}$$

Experiments



NSGAII:

- Computation time: 0.6 s,
- Population size: 200,
- Generation: 300.

ParCont:

- Computation time: 0.46 s,
- 534 parallelotopes.

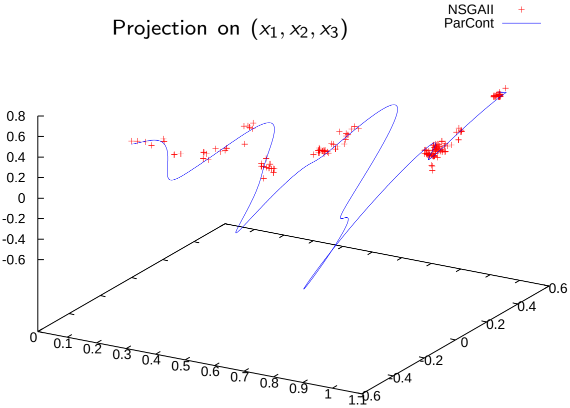
Experiments

LZ3 Modified [Li and Zhang, 2009]: n-dimensional bi-objective problem.

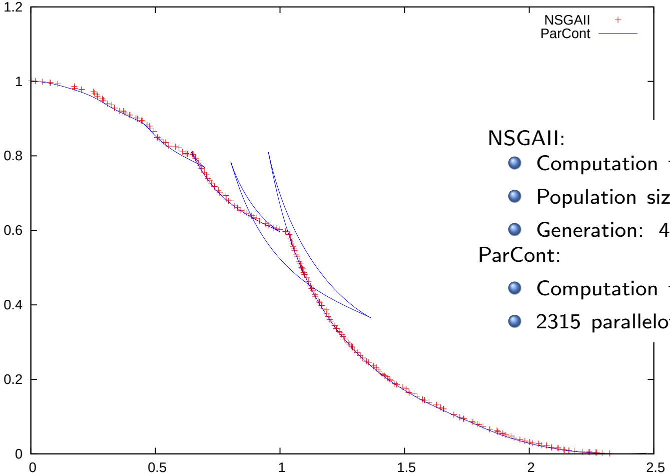
$$\left[\begin{array}{ll} \min & f_1(x) = x_1 + \frac{2}{n}\sum_{i=2}^{n}(x_i - 0.8x_1\cos(6\pi x_1 + \frac{i\pi}{n}))^2 \\ \min & f_2(x) = 1 - x_1^2 + \frac{2}{n}\sum_{i=2}^{n}(x_i - 0.8x_1\sin(6\pi x_1 + \frac{i\pi}{n}))^2 \\ \text{s.t} & x_1 \leq 1 \\ & x_1 \geq 0 \\ & x_i \leq 1, i = 2, \ldots, n \\ & x_i \geq -1, i = 2, \ldots, n \end{array}\right]$$

ParCont start at f_2^* with $n = 10$.

Experiments



Experiments



NSGAII:

- Computation time: 1.3 s,
- Population size: 200,
- Generation: 400.

ParCont:

- Computation time: 11.2 s,
- 2315 parallelotopes.