IRCCyN, Ecole Centrale de Nantes

Kinematic Analysis and singularities of lower-mobility parallel manipulators based on algebraic geometry techniques

9 June 2016

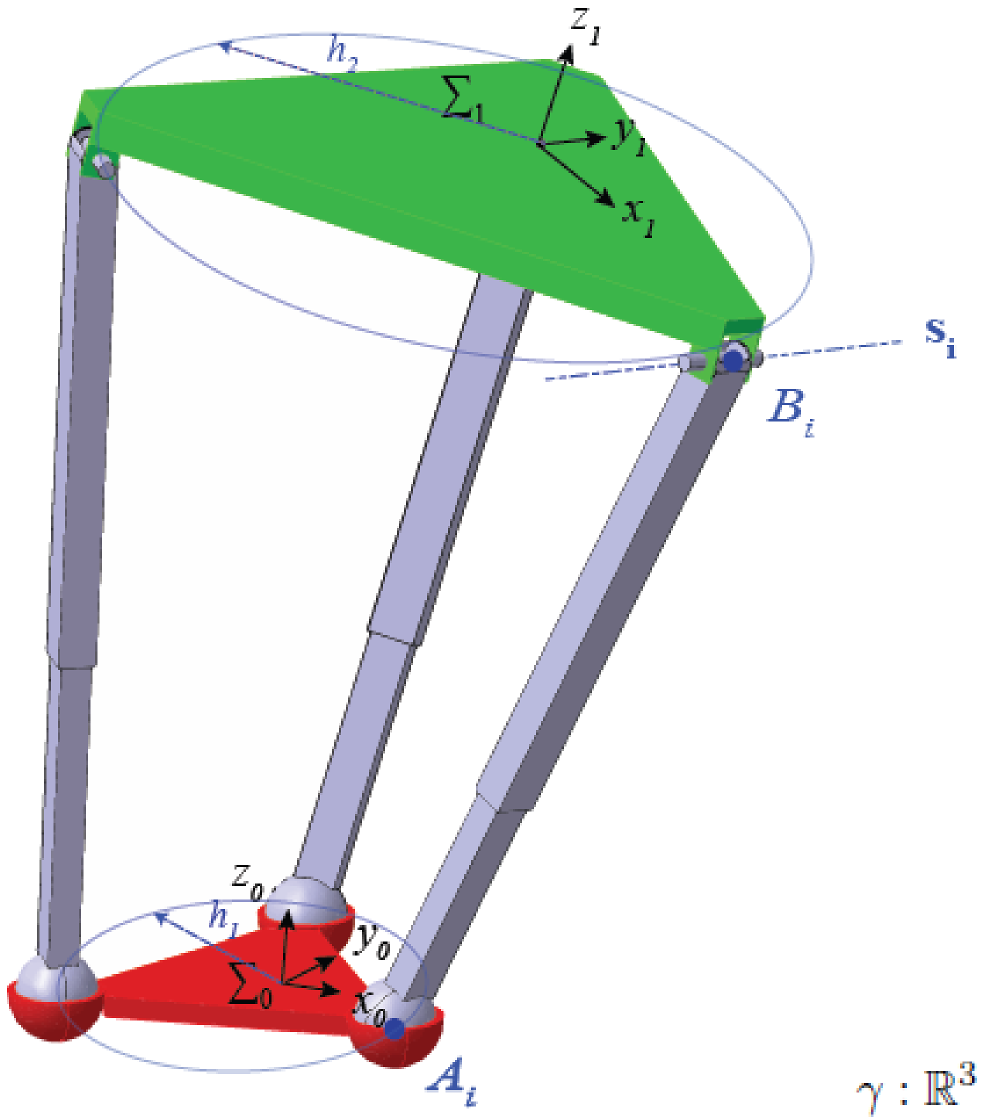
Abhilash NAYAK, PhD student, ECN

Supervisors : Stéphane CARO, Philippe WENGER

Contents

- Study’s kinematic mapping
- 3-SPR constraint equations
- Operation modes
- Singularities

3-SPR parallel manipulator



Study's kinematic mapping

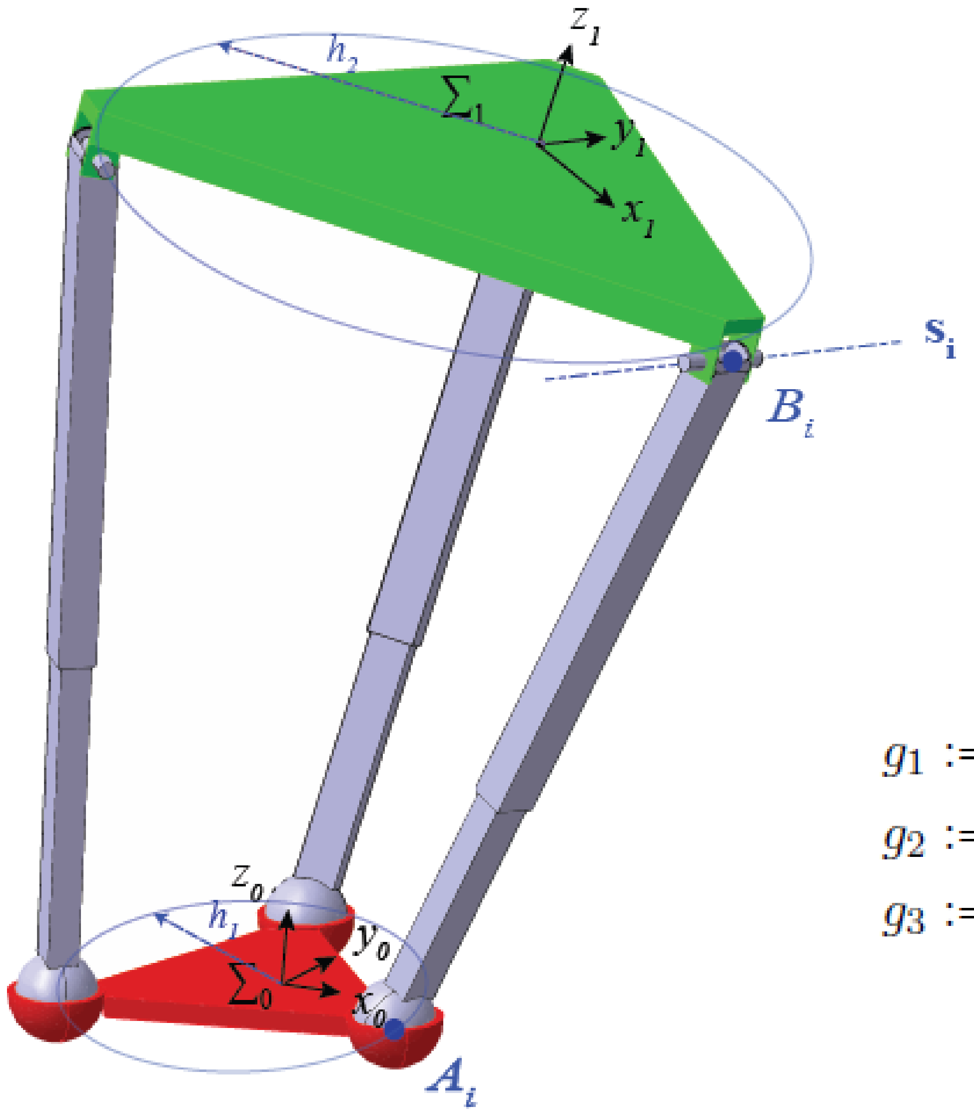
$$\mathbf{M} = \begin{bmatrix} x_0{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2 & \mathbf{0}_{3\times 1}^T \\ \mathbf{M}_T & \mathbf{M}_R \end{bmatrix}$$

$$\mathbf{M}_T = \begin{bmatrix} -2x_0y_1 + 2x_1y_0 - 2x_2y_3 + 2x_3y_2 \\ -2x_0y_2 + 2x_1y_3 + 2x_2y_0 - 2x_3y_1 \\ -2x_0y_3 - 2x_1y_2 + 2x_2y_1 + 2x_3y_0 \end{bmatrix}$$

$$\mathbf{M}_R = \begin{bmatrix} x_0{}^2 + x_1{}^2 - x_2{}^2 - x_3{}^2 & -2x_0x_3 + 2x_1x_2 & 2x_0x_2 + 2x_1x_3 \\ 2x_0x_3 + 2x_1x_2 & x_0{}^2 - x_1{}^2 + x_2{}^2 - x_3{}^2 & -2x_0x_1 + 2x_3x_2 \\ -2x_0x_2 + 2x_1x_3 & 2x_0x_1 + 2x_3x_2 & x_0{}^2 - x_1{}^2 - x_2{}^2 + x_3{}^2 \end{bmatrix}$$

$$\gamma : \mathbb{R}^3 \to \mathbb{R}^3,\ \ \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{d} \longrightarrow p \in \mathbb{P}^7$$

3-SPR parallel manipulator



Constraint equations

$$\mathbf{r}^0_{B_i} = \mathbf{M}\ \mathbf{r}^1_{B_i} \qquad \mathbf{s}^0_i = \mathbf{M}\ \mathbf{s}^1_i \qquad i = 1, 2, 3$$

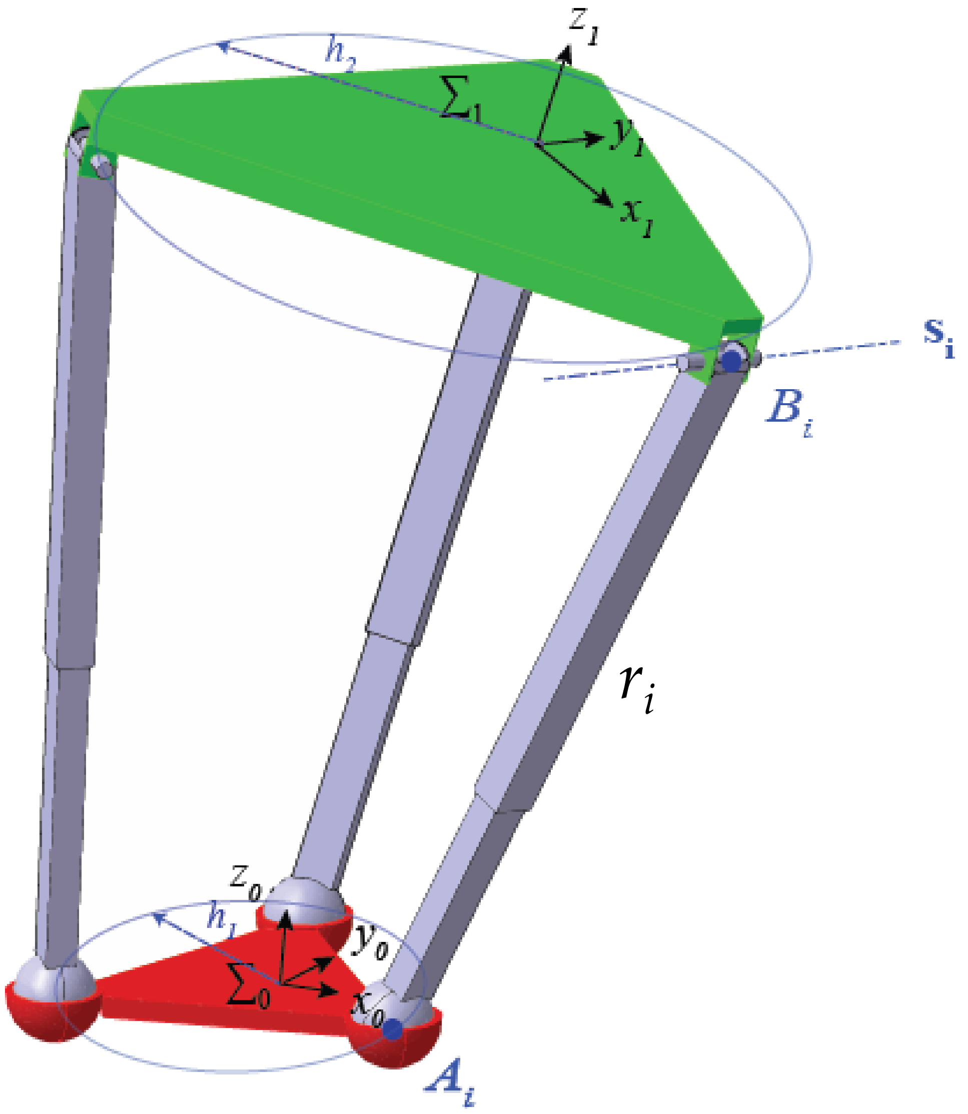
$$(\mathbf{r}^0_{B_i} - \mathbf{r}^0_{A_i})^T \mathbf{s}_i = 0$$

$$g_1 := x_0 x_3 = 0$$

$$g_2 := h_1 {x_1}^2 - h_1 {x_2}^2 - 2\, x_0 y_1 + 2\, x_1 y_0 + 2\, x_2 - 2\, x_3 y_2 = 0$$

$$g_3 := 2\, h_1 x_0 x_3 + h_1 x_1 x_2 + x_0 y_2 + x_1 y_3 - x_2 y_0 - x_3 y_1 = 0$$

3-SPR parallel manipulator



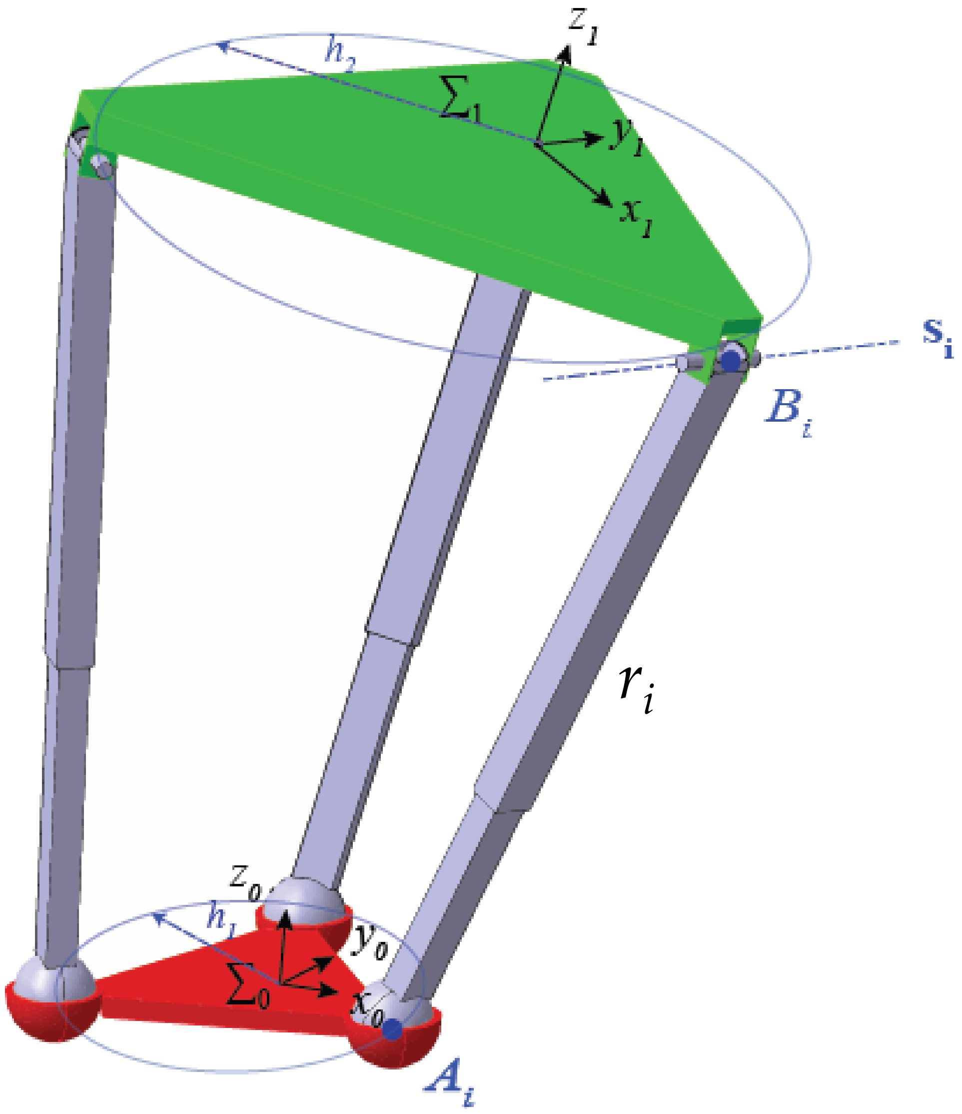
Constraint equations

$$||\,\mathbf{r}^0_{B_i} - \mathbf{r}^0_{A_i}\,||^2 = r_i^2$$

$$\begin{aligned}
g_4 := & -r_1^2 x_0^2 - r_1^2 x_1^2 - r_1^2 x_2^2 - r_1^2 x_3^2 + \left(h_1^2 - 2\,h_1 h_2 + h_2^2\right) x_0^2 \\
& + (4\,h_1 - 4\,h_2)\, x_0 y_1 + \left(h_1^2 - 2\,h_1 h_2 + h_2^2\right) x_1^2 + (-4\,h_1 + 4\,h_2)\, x_1 y_0 \\
& + \left(h_1^2 + 2\,h_1 h_2 + h_2^2\right) x_2^2 + (4\,h_1 + 4\,h_2)\, x_2 y_3 + \left(h_1^2 + 2\,h_1 h_2 + h_2^2\right) \\
& x_3^2 + (-4\,h_1 - 4\,h_2)\, x_3 y_2 + 4\,y_0^2 + 4\,y_1^2 + 4\,y_2^2 + 4\,y_3^2 = 0
\end{aligned}$$

$$\begin{aligned}
g_5 := & \; r_2^2 x_0^2 + r_2^2 x_1^2 + r_2^2 x_2^2 + r_2^2 x_3^2 + \left(-h_1^2 + 2\,h_1 h_2 - h_2^2\right) x_0^2 \\
& + (2\,h_1 - 2\,h_2)\, x_0 y_1 + \left(2\sqrt{3}h_1 - 2\sqrt{3}h_2\right) x_0 y_2 + \left(-h_1^2 - h_1 h_2 - h_2^2\right) \\
& x_1^2 + 2\sqrt{3}h_1 h_2 x_1 x_2 + (-2\,h_1 + 2\,h_2)\, x_1 y_0 + \left(-2\sqrt{3}h_1 - 2\sqrt{3}h_2\right) x_1 y_3 \\
& + \left(-h_1^2 + h_1 h_2 - h_2^2\right) x_2^2 + \left(-2\sqrt{3}h_1 + 2\sqrt{3}h_2\right) x_2 y_0 + (2\,h_1 + 2\,h_2) \\
& x_2 y_3 + \left(-h_1^2 - 2\,h_1 h_2 - h_2^2\right) x_3^2 + \left(2\sqrt{3}h_1 + 2\sqrt{3}h_2\right) x_3 y_1 \\
& + (-2\,h_1 - 2\,h_2)\, x_3 y_2 - 4\,y_0^2 - 4\,y_1^2 - 4\,y_2^2 - 4\,y_3^2 = 0
\end{aligned}$$

3-SPR parallel manipulator



Constraint equations

$$|| \mathbf{r}^0_{B_i} - \mathbf{r}^0_{A_i} ||^2 = r_i^2$$

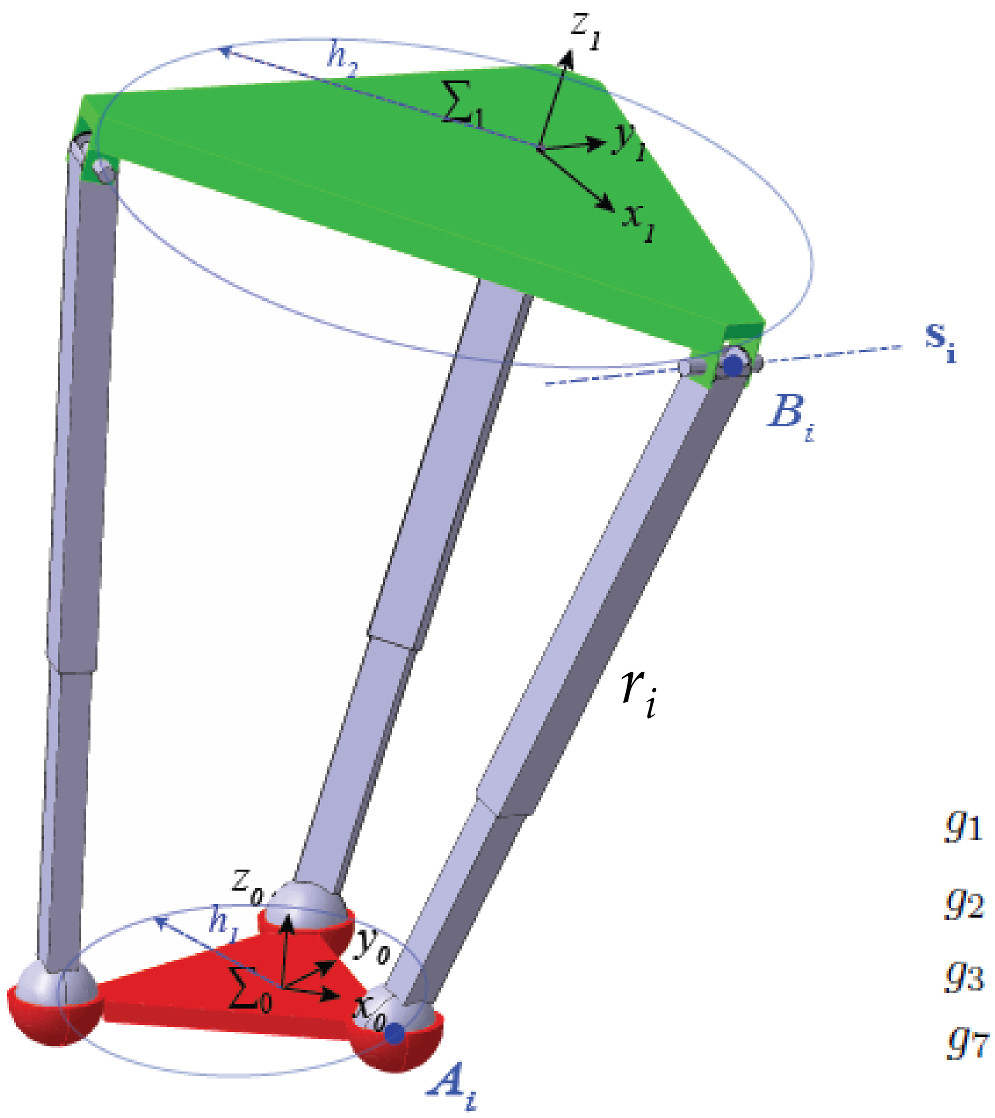
$$\begin{aligned} g_6 := & -r_3{}^2 x_0{}^2 - r_3{}^2 x_1{}^2 - r_3{}^2 x_2{}^2 - r_3{}^2 x_3{}^2 \left(h_1{}^2 - 2\,h_1 h_2 + h_2{}^2\right) x_0{}^2 \\ & + \left(-2\,h_1 + 2\,h_2\right) x_0 y_1 + \left(2\,\sqrt{3} h_1 - 2\,\sqrt{3} h_2\right) x_0 y_2 + \left(h_1{}^2 + h_1 h_2 + h_2{}^2\right) \\ & x_1{}^2 + 2\,\sqrt{3} h_1 h_2 x_1 x_2 + \left(2\,h_1 - 2\,h_2\right) x_1 y_0 + \left(-2\,\sqrt{3} h_1 - 2\,\sqrt{3} h_2\right) x_1 y_3 \\ & + \left(h_1{}^2 - h_1 h_2 + h_2{}^2\right) x_2{}^2 + \left(-2\,\sqrt{3} h_1 + 2\,\sqrt{3} h_2\right) x_2 y_0 + \left(-2\,h_1 - 2\,h_2\right) \\ & x_2 y_3 + \left(h_1{}^2 + 2\,h_1 h_2 + h_2{}^2\right) x_3{}^2 + \left(2\,\sqrt{3} h_1 + 2\,\sqrt{3} h_2\right) x_3 y_1 \\ & + \left(2\,h_1 + 2\,h_2\right) x_3 y_2 + 4\,y_0{}^2 + 4\,y_1{}^2 + 4\,y_2{}^2 + 4\,y_3{}^2 = 0 \end{aligned}$$

Study equation ; Normalization equation

$$g_7 := x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

$$g_8 := x_0{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2 - 1 = 0$$

3-SPR parallel manipulator



Operation modes

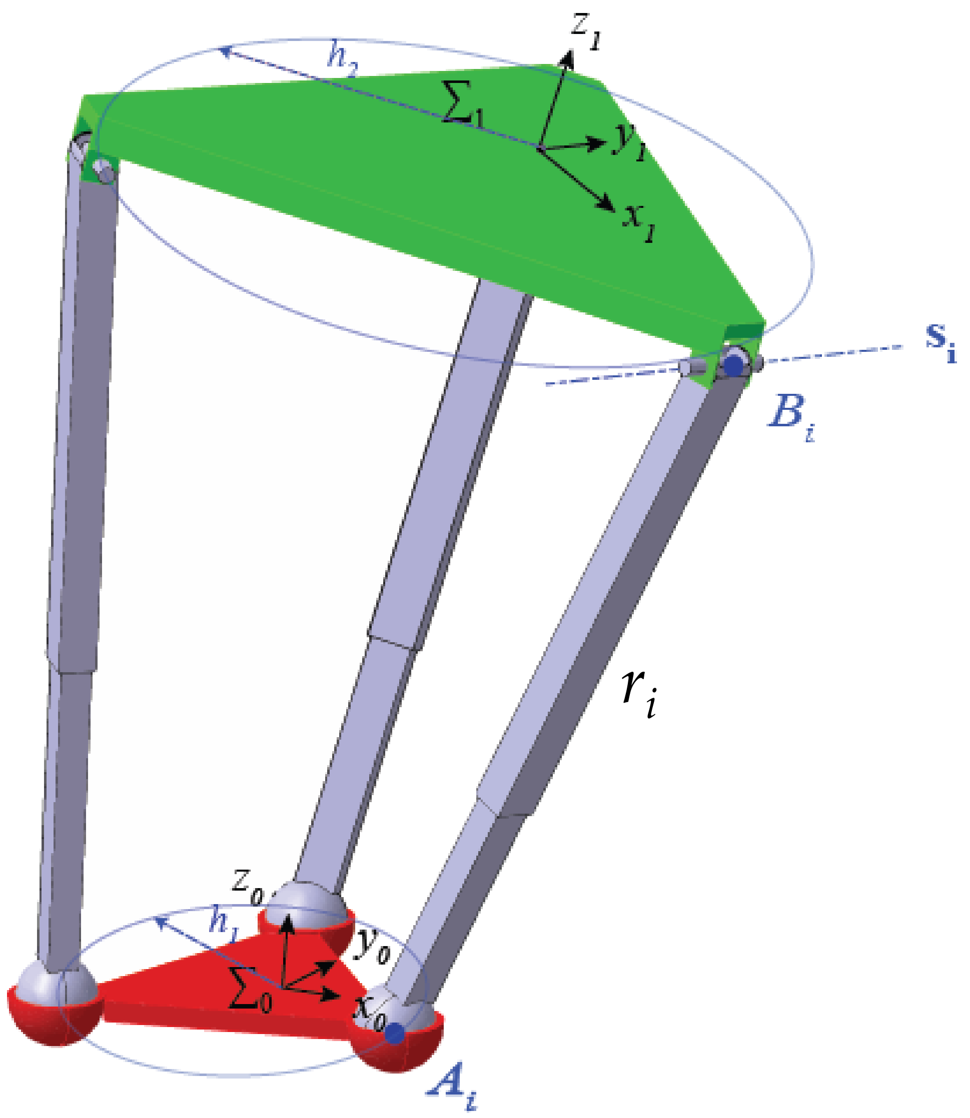
$$\mathcal{I} = < g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 >$$

Variables	Ring
$\{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\}$	$\mathbb{C}[h_1, h_2, r_1, r_2, r_3]$

$$\mathcal{J} = < g_1, g_2, g_3, g_7 >$$

$$g_1 := x_0 x_3 = 0$$
$$g_2 := h_1 {x_1}^2 - h_1 {x_2}^2 - 2\, x_0 y_1 + 2\, x_1 y_0 + 2\, x_2 - 2\, x_3 y_2 = 0$$
$$g_3 := 2\, h_1 x_0 x_3 + h_1 x_1 x_2 + x_0 y_2 + x_1 y_3 - x_2 y_0 - x_3 y_1 = 0$$
$$g_7 := x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

3-SPR parallel manipulator



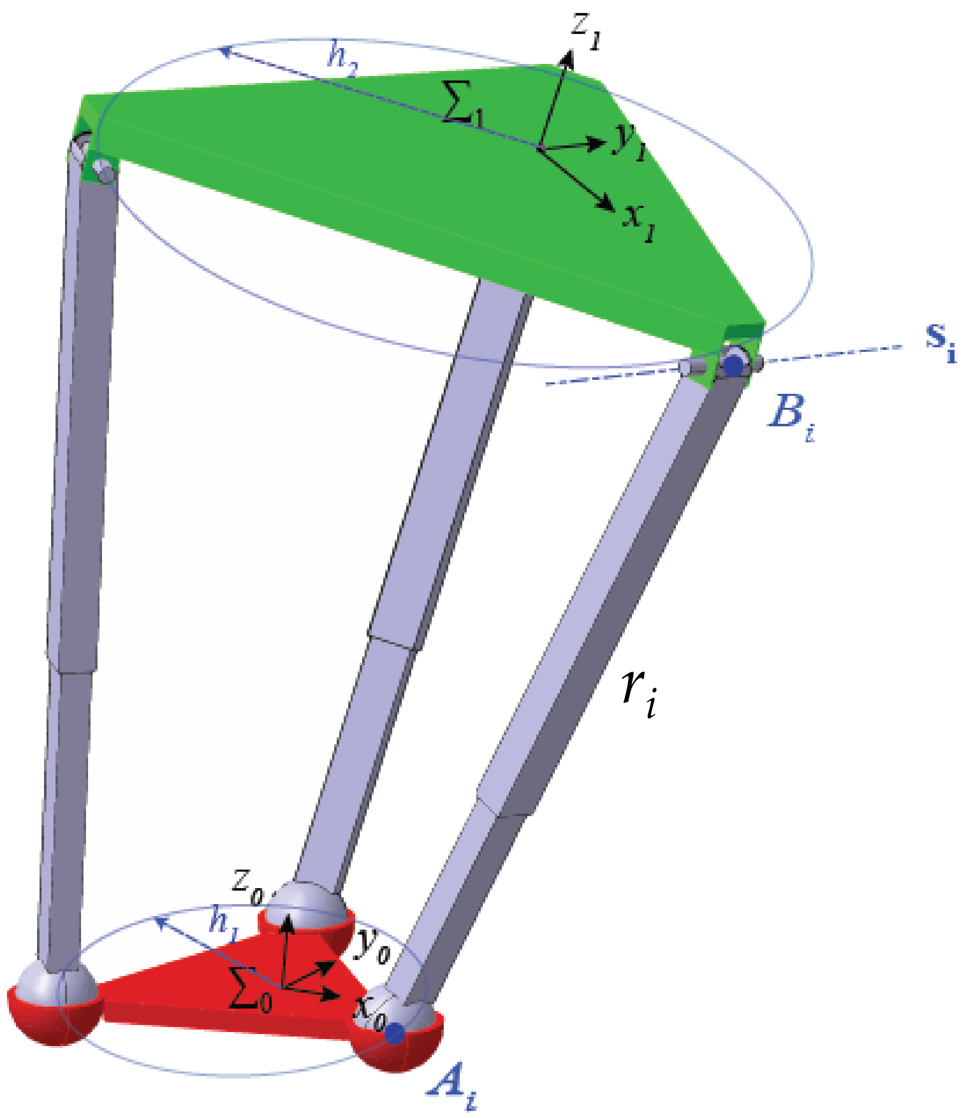
Operation modes $\mathcal{J} = < g_1, g_2, g_3, g_7 >$

$$\begin{aligned}\mathcal{J}_1 : \langle & x_0, x_1y_1 + x_2y_2 + x_3y_3, h_1x_1x_2 + x_1y_3 - x_2y_0 - x_3y_1, h_1{x_1}^2 - h_1{x_2}^2 \\ & + 2\,x_1y_0 + 2\,x_2y_3 - 2\,x_3y_2, h_1{x_2}^2y_2 + h_1x_2x_3y_3 - x_1y_1y_3 + x_2y_0y_1 \\ & + x_3{y_1}^2, h_1{x_2}^3 + {x_1}^2y_3 - 3\,x_1x_2y_0 - x_1x_3y_1 - 2\,{x_2}^2y_3 + 2\,x_2x_3y_2, \\ & h_1x_1x_2y_2 + h_1x_1x_3y_3 + h_1{x_2}^2y_1 - 2\,x_1y_0y_1 - 2\,x_2y_1y_3 + 2\,x_3y_1y_2, \\ & {h_1}^2{x_2}^2y_3 - h_1x_1y_0y_3 + h_1x_2{y_1}^2 - 3\,h_1x_2{y_2}^2 - h_1x_2{y_3}^2 - h_1x_3y_2y_3 \\ & - 2\,{y_0}^2y_3 - 2\,{y_1}^2y_3 - 2\,{y_2}^2y_3 - 2\,{y_3}^3, -{h_1}^2{x_2}^2y_0y_3 + {h_1}^2{x_2}^2y_1y_2 \\ & + {h_1}^2x_2x_3y_1y_3 + h_1x_1{y_0}^2y_3 - h_1x_1{y_1}^2y_3 + 3\,h_1x_2y_0{y_2}^2 + h_1x_2y_0{y_3}^2 \\ & + h_1x_3y_0y_2y_3 + h_1x_3{y_1}^3 + 2\,{y_0}^3y_3 + 2\,y_0{y_1}^2y_3 + 2\,y_0{y_2}^2y_3 + 2\,y_0{y_3}^3 \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{J}_2 : \langle & x_3, x_0y_0 + x_1y_1 + x_2y_2, h_1x_1x_2 + x_0y_2 + x_1y_3 - x_2y_0, h_1{x_1}^2 \\ & - h_1{x_2}^2 - 2\,x_0y_1 + 2\,x_1y_0 + 2\,x_2y_3, h_1{x_2}^3 + x_0x_1y_2 + 2\,x_0x_2y_1 \\ & + {x_1}^2y_3 - 3\,x_1x_2y_0 - 2\,{x_2}^2y_3, {h_1}^2{x_2}^2y_0 - h_1x_1{y_0}^2 - h_1x_1{y_2}^2 \\ & - h_1x_2y_0y_3 - 3\,h_1x_2y_1y_2 - 2\,{y_0}^3 - 2\,y_0{y_1}^2 - 2\,y_0{y_2}^2 - 2\,y_0{y_3}^2 \rangle\end{aligned}$$

$$\mathcal{J}_3 : \langle x_0, x_1, x_2, x_3 \rangle$$

3-SPR parallel manipulator



Operation modes $\mathcal{J} =< g_1, g_2, g_3, g_7 >$

$$\mathcal{J} = \bigcap_{i=1}^{3} \mathcal{J}_i \quad or \quad V(\mathcal{J}) = \bigcup_{i=1}^{3} V(\mathcal{J}_i)$$

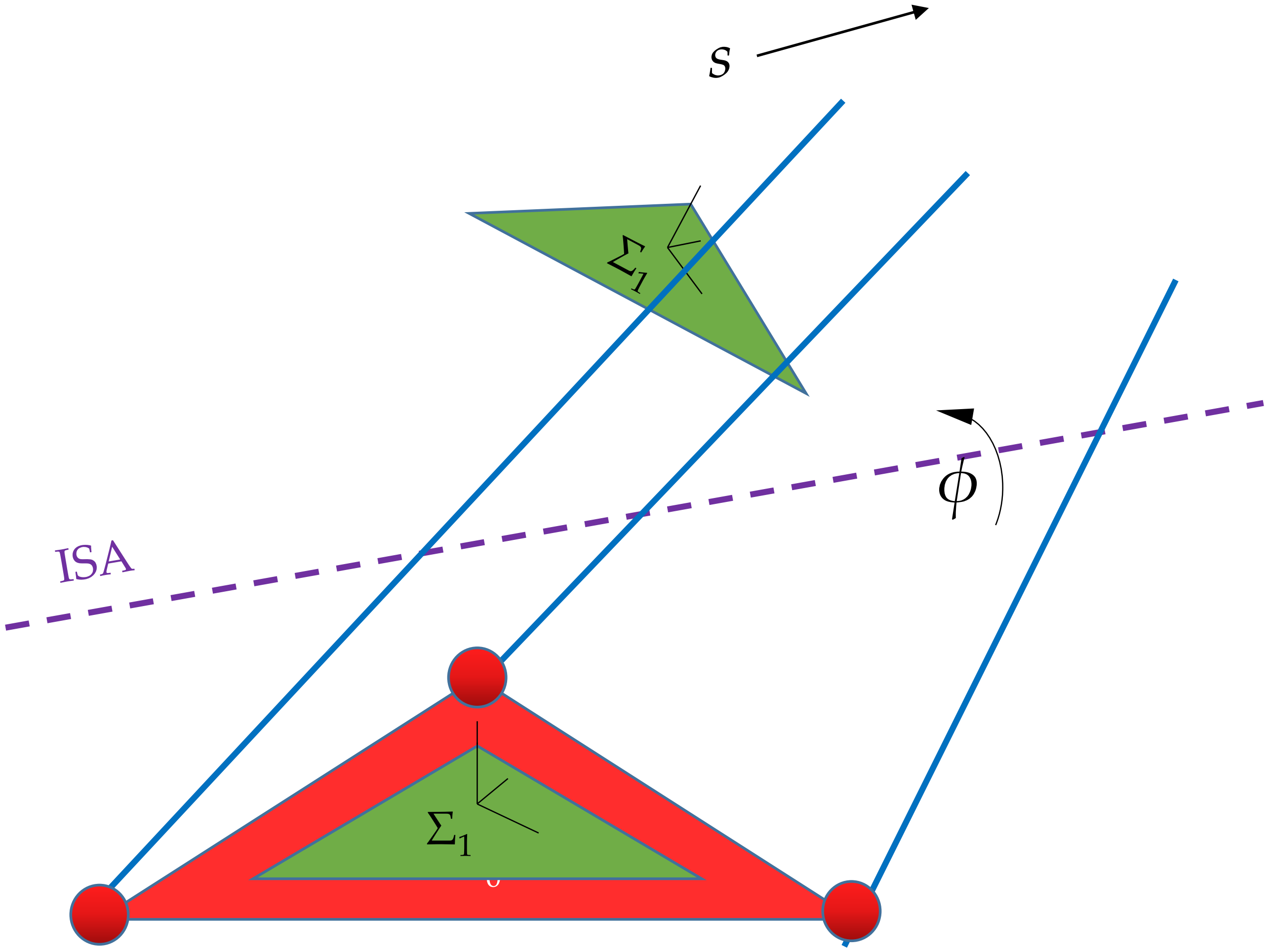
Direct Kinematics

$$\mathcal{K}_i = \mathcal{J}_i \cup \langle g_4, g_5, g_6, g_8 \rangle \quad i = 1, 2$$

$\mathbb{C}[h_1, h_2, r_1, r_2, r_3]$

Gröebner Basis : 16 solutions

3-SPR parallel manipulator : Operation modes



Instantaneous screw axis

Plücker-coordinates of the ISA

$$\begin{aligned} p_{01} &= (-x_1^2 - x_2^2 - x_3^2)x_1, & p_{23} &= x_0y_0x_1 - (-x_1^2 - x_2^2 - x_3^2)y_1 \\ p_{02} &= (-x_1^2 - x_2^2 - x_3^2)x_2, & p_{31} &= x_0y_0x_2 - (-x_1^2 - x_2^2 - x_3^2)y_2 \\ p_{03} &= (-x_1^2 - x_2^2 - x_3^2)x_3, & p_{12} &= x_0y_0x_3 - (-x_1^2 - x_2^2 - x_3^2)y_3 \end{aligned}$$

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$$

Normalized Study-parameters

$$cos(\frac{\phi}{2}) = x_0 \quad ; \quad s = \frac{2y_0}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

3-SPR parallel manipulator : Operation modes

Operation mode 1 : $\mathcal{K}_1$

$$\boxed{x_0 = 0}$$

180^0 displacement of the moving platform w.r.t the fixed base about the ISA.

$$cos(\frac{\phi}{2}) = x_0 \quad ; \quad s = \frac{2y_0}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

Operation mode 2 : $\mathcal{K}_2$.

$$\boxed{x_3 = 0}$$

ISA is always parallel to the x_0y_0-plane

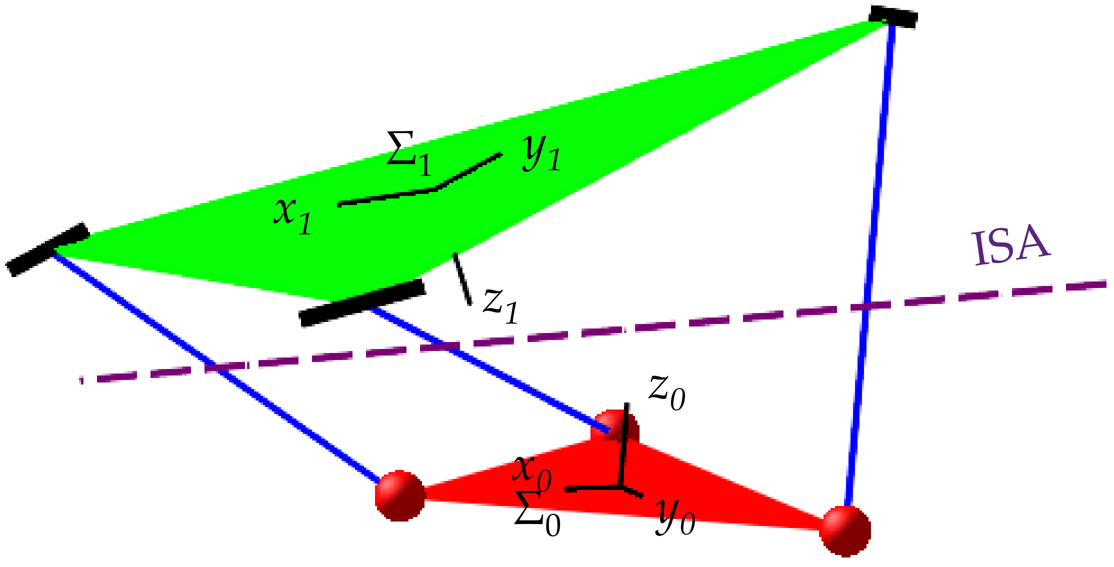
$$\begin{aligned}
p_{01} &= (-x_1^2 - x_2^2 - x_3^2)x_1, \quad & p_{23} &= x_0y_0x_1 - (-x_1^2 - x_2^2 - x_3^2)y_1\\
p_{02} &= (-x_1^2 - x_2^2 - x_3^2)x_2, \quad & p_{31} &= x_0y_0x_2 - (-x_1^2 - x_2^2 - x_3^2)y_2\\
p_{03} &= (-x_1^2 - x_2^2 - x_3^2)x_3, \quad & p_{12} &= x_0y_0x_3 - (-x_1^2 - x_2^2 - x_3^2)y_3
\end{aligned}$$

3-SPR parallel manipulator : Operation modes

Operation mode 1 : $\mathcal{K}_1$

$\boxed{x_0 = 0}$

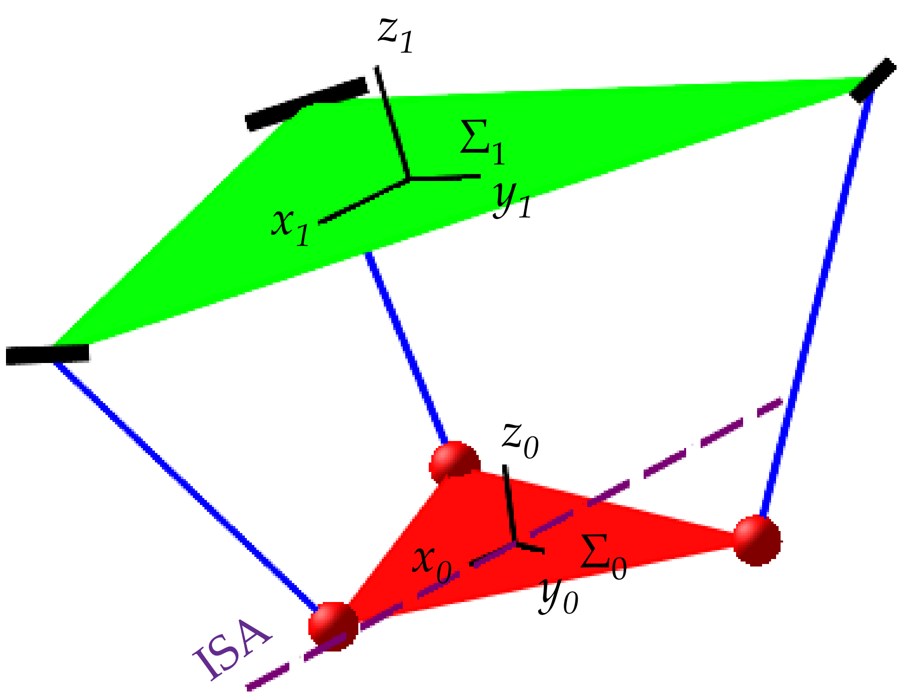
$h_1 = 1$, $h_2 = 2$, $r_1 = 1.8$, $r_2 = 4.2$, $r_3 = 2$



Operation mode 2 : $\mathcal{K}_2$.

$\boxed{x_3 = 0}$

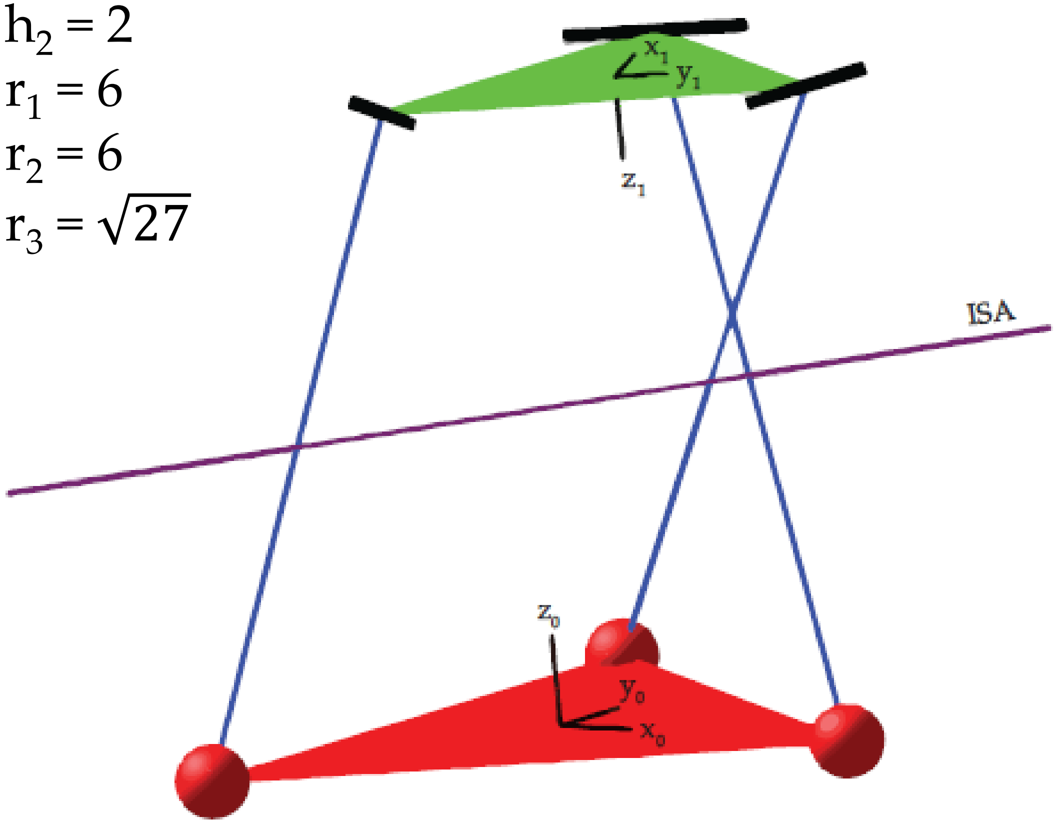
$h_1 = 1$, $h_2 = 2$, $r_1 = 1.8$, $r_2 = 1.7$, $r_3 = 2$



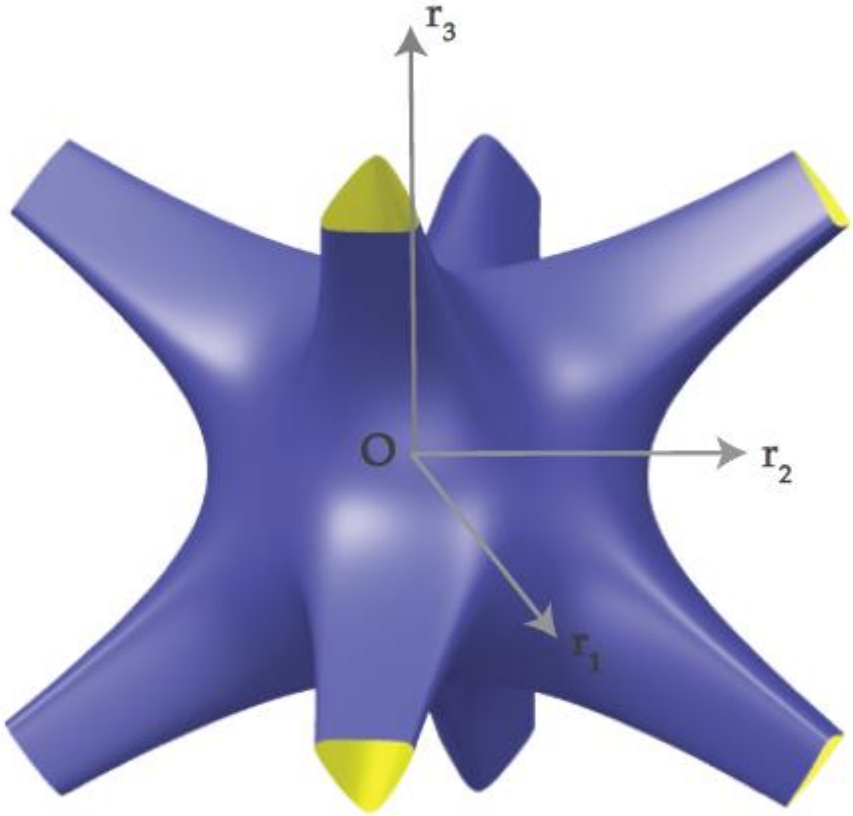
3-SPR parallel manipulator : Singularities

Constraint singularity (Transition mode) : $x_0 = x_3 = 0$

$h_1 = 1$
$h_2 = 2$
$r_1 = 6$
$r_2 = 6$
$r_3 = \sqrt{27}$



Singular pose



Joint space

3-SPR parallel manipulator : Singularities

Jacobian matrix $$\mathrm{J}_i = \left(\frac{\partial g_j}{\partial x_k}, \frac{\partial g_j}{\partial y_k} \right) \text{ where } \quad i = 1, 2\ ; \quad j = 1, ..., 8\ ; \quad k = 0, ..., 3$$

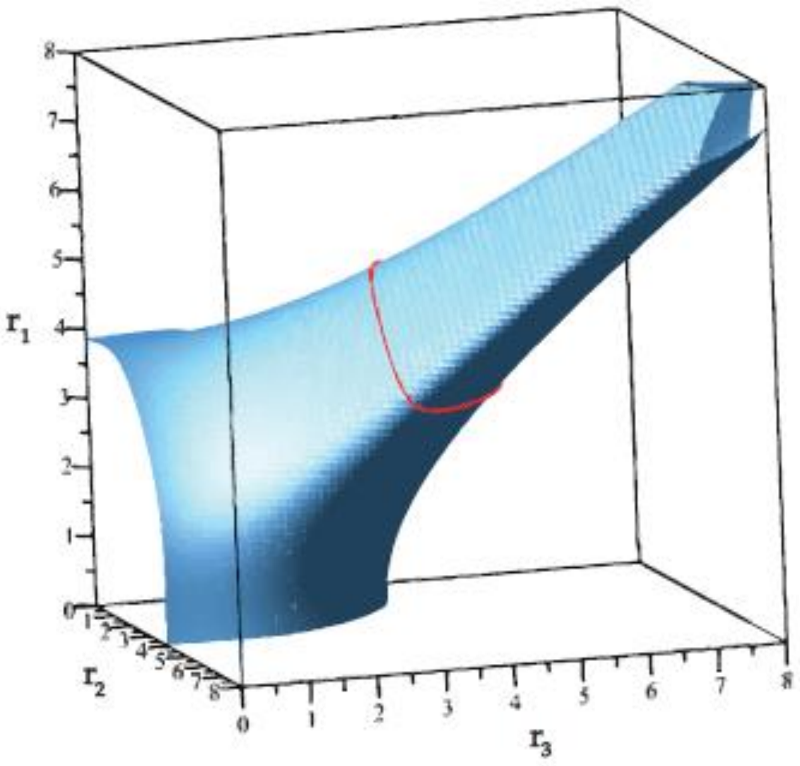
$$S_i{:}det(J_i)$$

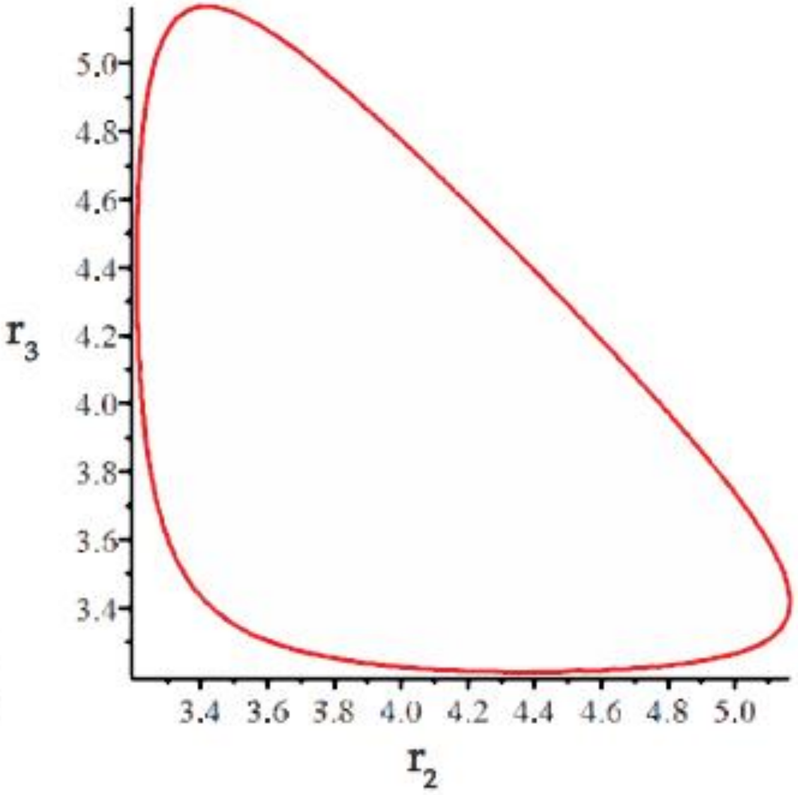
$$S_1 : x_3 \cdot p^7(x_1, x_2, x_3, y_0, y_1, y_2, y_3) = 0$$
$$S_2 : x_0 \cdot p^7(x_0, x_1, x_2, y_0, y_1, y_2, y_3) = 0$$

3-SPR parallel manipulator : Singularities

Constraint singularity in joint space : $x_0 = x_3 = 0$

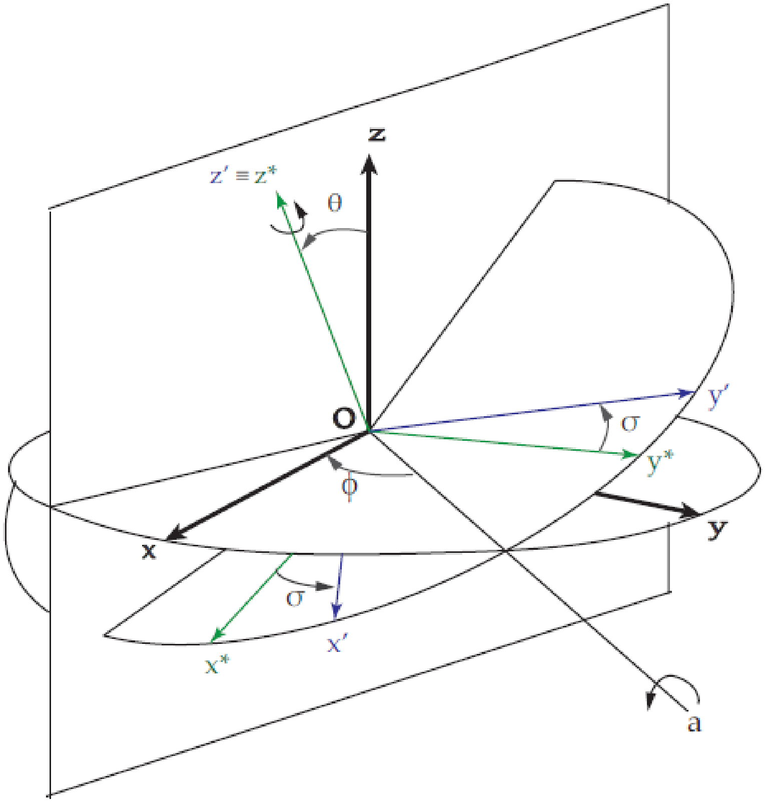
$$
\begin{aligned}
S_c : {} & r_1^8 - 2 r_1^6 r_2^2 - 2 r_1^6 r_3^2 + 3 r_1^4 r_2^4 + 3 r_1^4 r_3^4 - 2 r_1^2 r_2^6 - 2 r_1^2 r_3^6 + r_2^8 \\
& - 2 r_2^6 r_3^2 + 3 r_2^4 r_3^4 - 2 r_2^2 r_3^6 + r_3^8 - 96 r_1^6 + 144 r_1^4 r_2^2 + 144 r_1^4 r_3^2 \\
& + 144 r_1^2 r_2^4 - 576 r_1^2 r_2^2 r_3^2 + 144 r_1^2 r_3^4 - 96 r_2^6 + 144 r_2^4 r_3^2 \\
& + 144 r_2^2 r_3^4 - 96 r_3^6 + 2430 r_1^4 - 2430 r_1^2 r_2^2 - 2430 r_1^2 r_3^2 + 2430 r_2^4 \\
& - 2430 r_2^2 r_3^2 + 2430 r_3^4 - 273375 = 0
\end{aligned}
$$





3-SPR parallel manipulator : Singularities

Other singularities in orientation workspace



$$x_0 = \cos(\frac{\theta}{2})\cos(\frac{\sigma}{2})$$
$$x_1 = \sin(\frac{\theta}{2})\cos(\phi - \frac{\sigma}{2})$$
$$x_2 = \sin(\frac{\theta}{2})\sin(\phi - \frac{\sigma}{2})$$
$$x_3 = \cos(\frac{\theta}{2})\sin(\frac{\sigma}{2})$$

Tilt and torsion angles

$$\mathbf{M}' = \begin{bmatrix} x_0^2+x_1^2+x_2^2+x_3^2 & 0 & 0 & 0 \\ X & {x_0}^2+{x_1}^2-{x_2}^2-{x_3}^2 & -2\,x_0x_3+2\,x_1x_2 & 2\,x_0x_2+2\,x_1x_3 \\ Y & 2\,x_0x_3+2\,x_1x_2 & {x_0}^2-{x_1}^2+{x_2}^2-{x_3}^2 & -2\,x_0x_1+2\,x_3x_2 \\ Z & -2\,x_0x_2+2\,x_1x_3 & 2\,x_0x_1+2\,x_3x_2 & {x_0}^2-{x_1}^2-{x_2}^2+{x_3}^2 \end{bmatrix}$$

Modified matrix of Study’s kinematic mapping

3-SPR parallel manipulator : Singularities

Singularities in each operation mode (in orientation workspace)

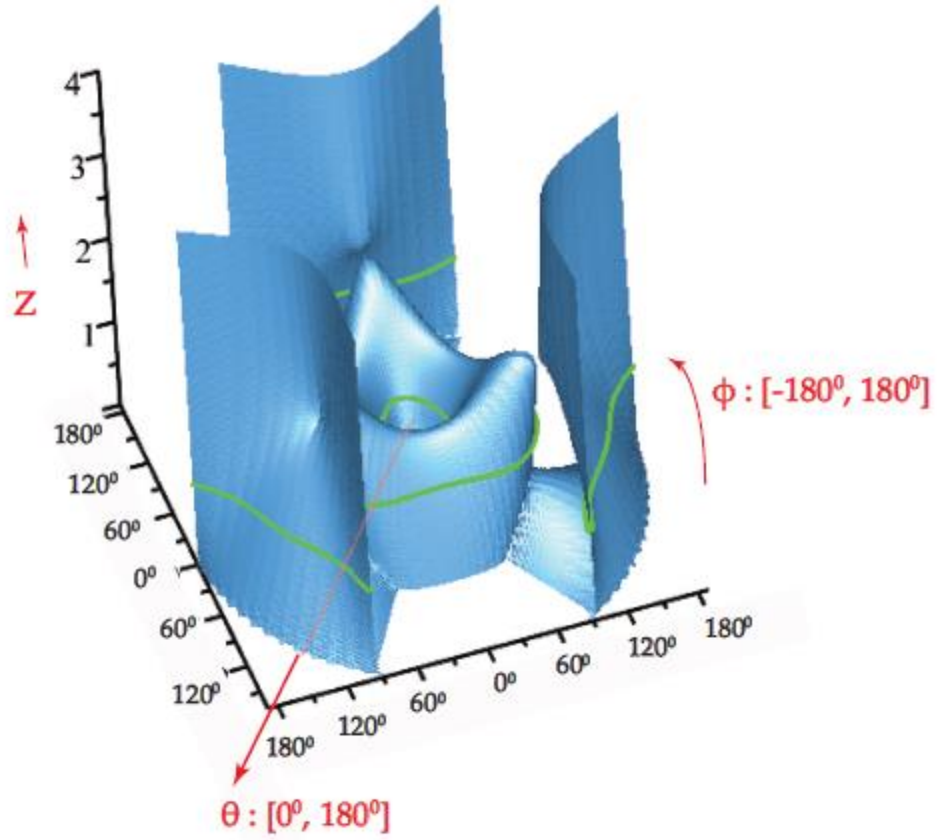
Operation mode 1 :

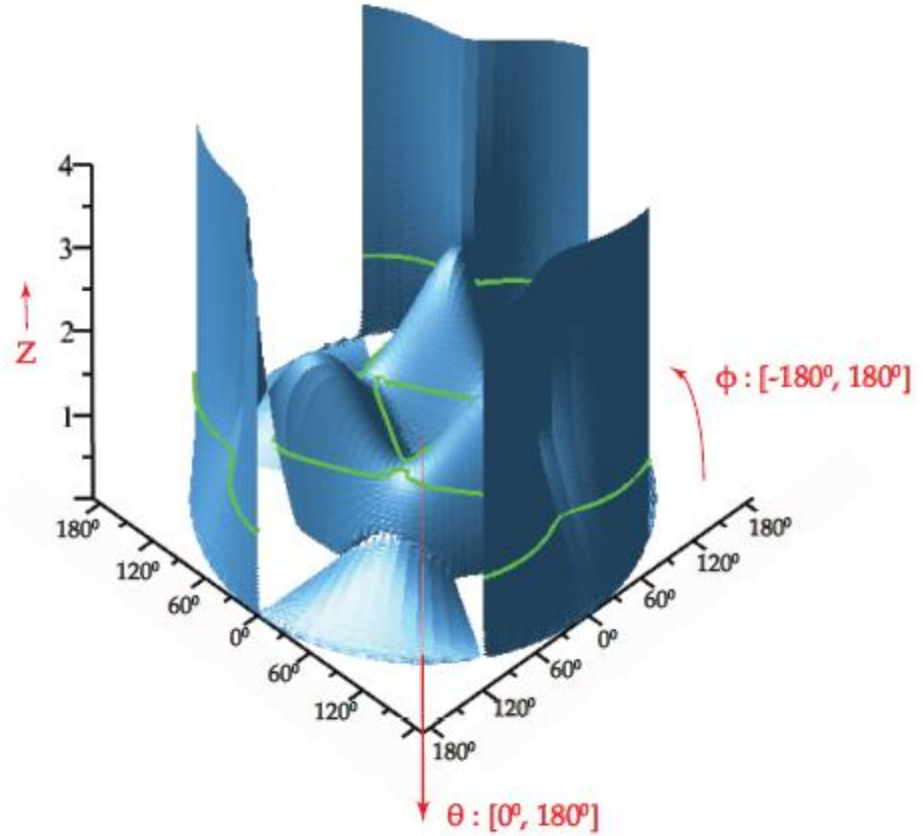
$x_0 = 0$: $\sigma = 180^0$

$h_1 = 1$
$h_2 = 2$

Operation mode 2 :

$x_3 = 0$: $\sigma = 0^0$





3-SPR parallel manipulator : Singularities

Singularities in each operation mode (in orientation workspace)

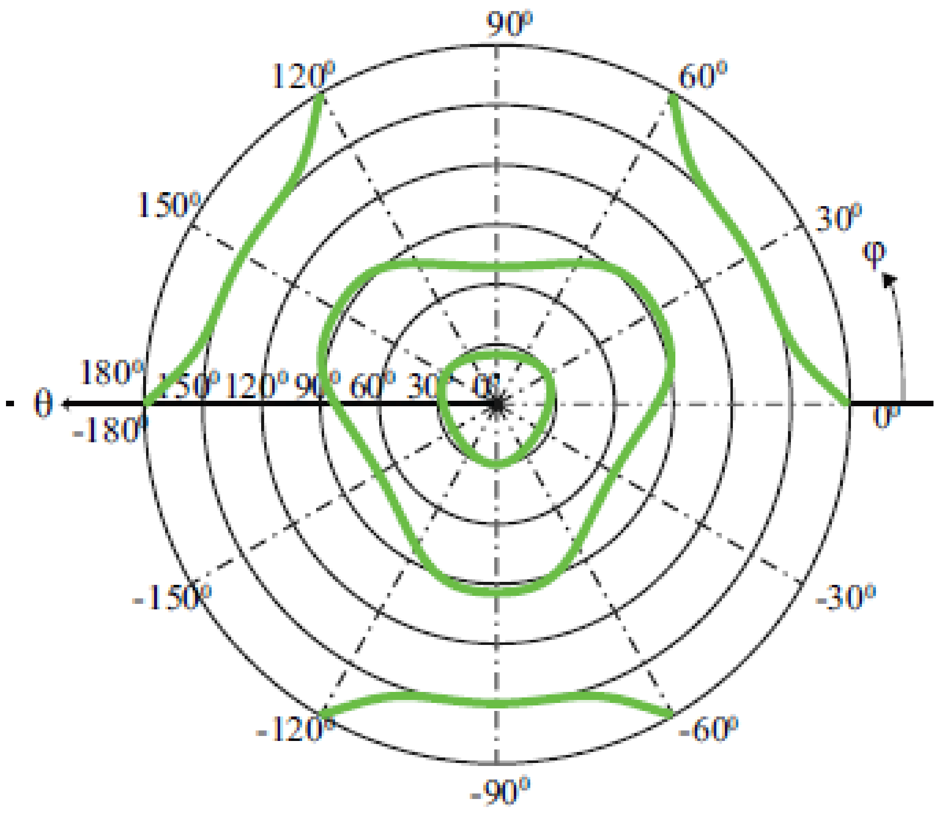
Operation mode 1 :

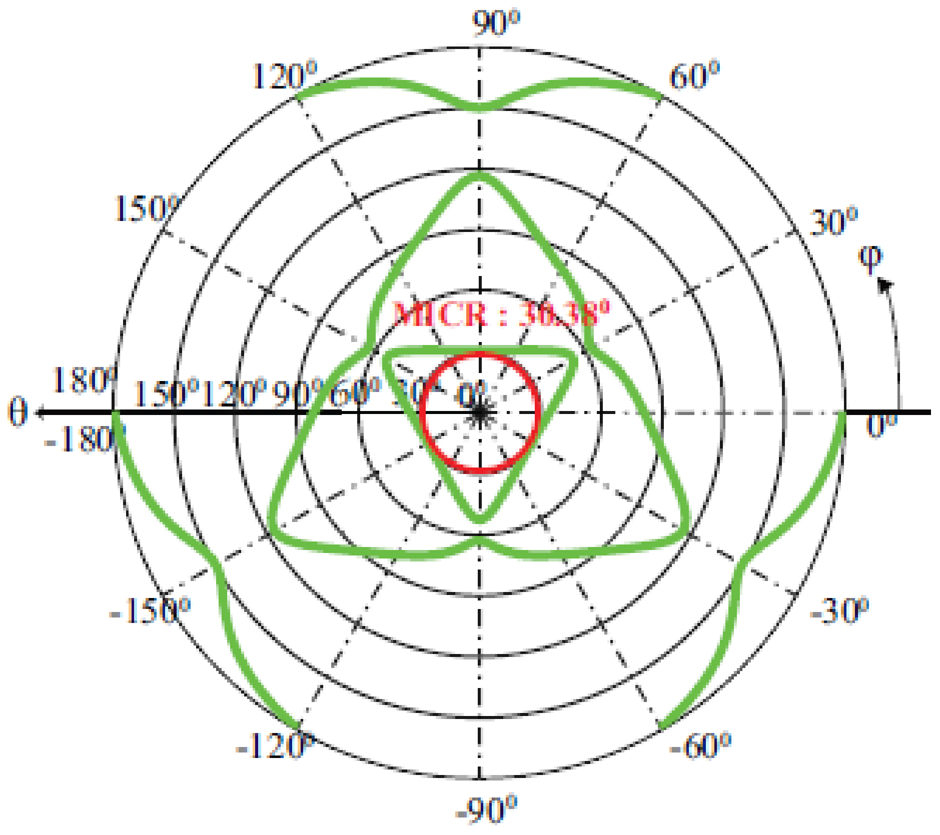
$x_0 = 0$: $\sigma = 180^0$

$h_1 = 1$
$h_2 = 2$
$Z = 1$

Operation mode 2 :

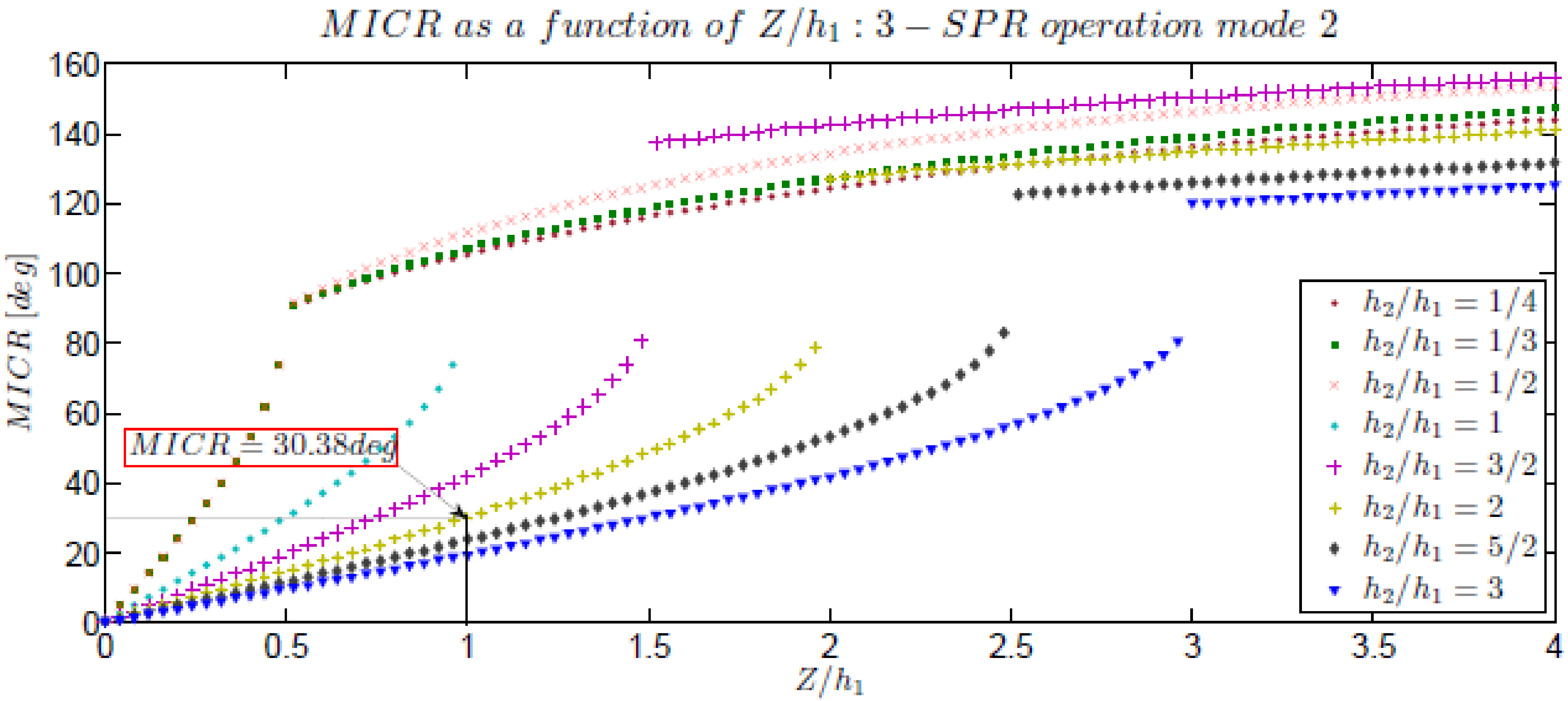
$x_3 = 0$: $\sigma = 0^0$





3-SPR parallel manipulator : Singularities

Maximum Inscribed Circle Radius (MICR)



Questions

- Choice of equations to solve direct kinematics in Gröebner Basis.
- Maximum Inscribed shape for any Z.