A parametric Kantorovich theorem with application to tolerance synthesis [GCC15]

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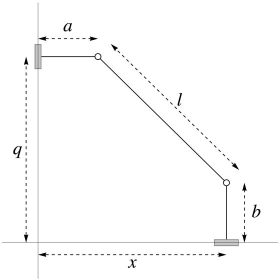
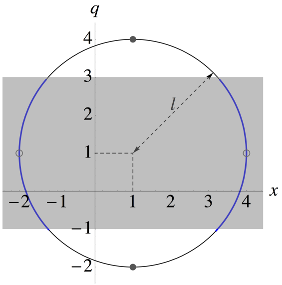
Motivation





$$\begin{array}{c}(x-1)^2+(q-1)^2=3^2\\ -3\le x\le 5\wedge -1\le q\le 3\end{array}\qquad (1)$$

Parallel robot

- Kinematic model: system of equations $f(x,q,p)=0$ with
  - $x\in\mathbb{R}^n$ the pose
  - $q\in\mathbb{R}^m$ the command (often $m=n$)
  - $p\in\mathbb{R}^q$ uncertainties on system parameters
  - $f(x,q,p)\in\mathbb{R}^n$ (as many equations as pose coordinates)
- Nominal workspace: $\mathcal{W}:=\{(x,q)\in\mathbb{R}^n\times\mathbb{R}^m : f(x,q,0)=0, g(x,q)\le 0\}$

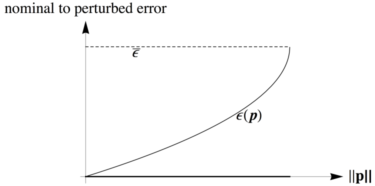
Motivation

Direct kinematic problem

- q and p fixed, compute x such that $f(x, q, p) = 0$
  - $p = 0 \Longrightarrow x$ is a nominal solution
  - $p \neq 0 \Longrightarrow x$ is a perturbed solution
- Square system of equations

Tolerance analysis

- Given $\Delta \geq ||p||$
- Find $\bar{\epsilon}$ maximal distance between x and x_p in the workspace



$\longleftarrow$ Typical result

- $\epsilon(p)$ in addition to $\bar{\epsilon}$
- Too large p make no sense

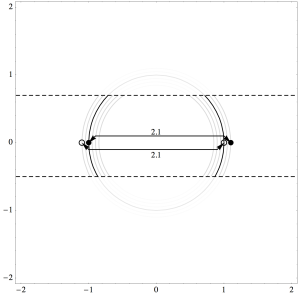
$\Leftrightarrow$ Is Δ small enough?

$\Rightarrow$ Need to compute a domain for p too

Motivation

Solved by global optimization?

$$\max_{\substack{f(x,q,0)=0,\ g(x,q)\leq 0\\ f(x_p,q,p)=0,\ \|p\|\leq\Delta}} \|x - x_p\| \tag{2}$$



- Make no sense:
  Incorrect nominal↔perturbed association
- Need initial domain for x_p
  - Large enough to contain x_p
  - Small enough to separate different solutions for fixed q

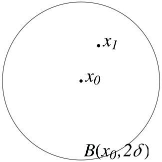
$\Longleftrightarrow$ Compute $\overline{\epsilon}$

$\Rightarrow$ Need a local analysis around nominal solutions

Kantorovich Theorem (simplified version)

Solve $f(x) = 0$

- Given x_0 compute the first Newton step $x_1 = x_0 + Df(x_0)^{-1} f(x_0)$
- Kantorovich constants χ, δ and λ
  - $\chi \geq \|Df(x_0)^{-1}\|$
  - $\delta \geq \|Df(x_0)^{-1} f(x_0)\| = \|x_1 - x_0\|$
  - λ Lipschitz constant for Df inside $B(x_0, 2\delta^+)$



- Domain for Lipchitz is known when δ is known
- Can be computed using second order derivatives:

$$\lambda \geq \max_{x \in B(x_0, 2\delta^+)} \max_i \sum_{j,k} \left| \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x) \right|$$

Kantorovich theorem

- χ conditioning – δ closeness linearized solution – λ nonlinearities
- $2\chi\delta\lambda \leq 1$ implies
  - $\exists x \in \overline{B}(x_0, t^*)$ with $t^*(\chi, \delta, \lambda) = \frac{1-\sqrt{1-2\delta\lambda\chi}}{\lambda\chi} \in [\delta, 2\delta]$
  - The solution is unique inside $\overline{B}(x_0, 2\delta)$

Application to sensitivity analysis: Crude upper bound

Classical trick

- Bound on $\|x - x_p\|$ is sought

$\Rightarrow$ Start Newton iterate at a nominal solution for solving the perturbed problem

  - x_0 satisfies $f(x_0, q, 0) = 0$
  - Kantorovich constants are computed for solving $f(x, q, p) = 0$

  $\Rightarrow$ $t^* \equiv$ upper bound nominal/perturbed solution

Evaluation Kantorovich constants using global optimization

- Don't need to compute every x_0 to evaluate Kantorovich constants !

$\Rightarrow$ Worst case constants for all q and p holds for every nominal solution

$$\chi \geq \max_{\substack{f(x,q,0)=0 \\ g(x,y)\leq 0 \\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1}\| \qquad \delta_0 \geq \max_{\substack{f(x,q,0)=0 \\ g(x,y)\leq 0 \\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1} f(x,q,p)\|$$

- Lipschitz constant: worst case inside $\overline{B}(x_0, (2\delta_0)^+)$ for all $(x_0, q) \in \mathcal{W}$

$2\chi\delta_0\lambda \leq 1$ implies for all $\|p\| \leq \Delta$
every nominal solution has a perturbed solution distant of at most $2\delta_0$

Application to sensitivity analysis: Sharp upper bound

Dependence of the first Newton step wrt $\|p\|$

- $f(x_0, q, p) = f(x_0, q, 0) + D_p f(x_0, q, 0)p + z = D_p f(x_0, q, 0)p + z$
- $\|z\| \leq \frac{1}{2}\mu\|p\|^2$ where μ is a Lipschitz constant for $D_p f$

  (for the whole workspace and all p such that $\|p\| \leq \Delta$)

$\Rightarrow$ $\|\Gamma_0\, f(x_0, q, p)\| \leq \|\Gamma_0\, D_p f(x_0, q, 0)\|\|p\| + \frac{1}{2}\chi\mu\|p\|^2$

- Worst case in workspace:

$$\delta_1(p) = \gamma\|p\| + \frac{1}{2}\chi\mu\|p\|^2 \quad \text{with}$$

$$\gamma \geq \max_{\substack{f(x,q,0)=0 \\ g(x,y)\leq 0 \\ \|p\|\leq\Delta}} \|D_x f(x, q, p)^{-1} D_p f(x, q, 0)\|$$

Second trick

- Kantorovich applies if $2\chi\lambda\,\delta(p) \leq 1$ with $\delta(p) := \min\{\delta_0, \delta_1(p)\}$

$\Rightarrow$ Choose the perturbation domain so that it applies:

$$\mathcal{P} := \{p \in \mathbb{R}^q : \|p\| \leq \Delta\ ,\ 2\chi\lambda\delta(p) \leq 1\}$$

Parametric Kantorovich theorem

Parametric Kantorovich constants χ, δ, γ, λ and μ

$$\chi \geq \max_{\substack{f(x,q,0)=0\\ g(x,y)\leq 0\\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1}\|$$

$$\delta_0 \geq \max_{\substack{f(x,q,0)=0\\ g(x,y)\leq 0\\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1} f(x,q,p)\|$$

$$\gamma \geq \max_{\substack{f(x,q,0)=0\\ g(x,y)\leq 0\\ \|p\|\leq\Delta}} \|D_x f(x,q,p)^{-1} D_p f(x,q,0)\|$$

- $D_x f(x,q,p)$ singular for some $(x,q,p) \Rightarrow \chi = +\infty$
- No nominal solution in $\mathcal{W} \Rightarrow$ infeasible problems

$$\forall (x_0,q)\in\mathcal{W}, \forall p\in \overline{B}_\Delta, \forall x',x''\in \overline{B}(x_0,(2\delta_0)^+), \|D_x f(x',q,p)-D_x f(x'',q,p)\| \leq \lambda\|x'-x''\|$$

$$\forall (x,q)\in\mathcal{W}, \forall p',p''\in \overline{B}_\Delta, \|D_p f(x,q,p')-D_p f(x,q,p'')\| \leq \mu\|p'-p''\|$$

Statement

$\delta_1(p)=\gamma\|p\|+\frac{1}{2}\chi\mu\|p\|^2$, $\delta(p)=\min\{\delta_0,\delta_1(p)\}$, $\mathcal{P}=\{p\in\mathbb{R}^q : \|p\|\leq\Delta\ ,\ 2\chi\lambda\delta(p)\leq 1\}$.

$$\forall p\in\mathcal{P}, \forall (x,q)\in\mathcal{W}, \exists! x_p\in B(x,\overline{\epsilon}),\ f(x_p,q,p)=0$$

with $\overline{\epsilon}=\min\{2\delta,\frac{1}{\chi\lambda}\}$. Furthermore, $\|x-x_p\|\leq\epsilon(p):=t^*(\chi,\delta(p),\lambda)$

The PRRP robot

Description

- $f(x, q, p) = (x - a - p_1)^2 + (q - b - p_2)^2 - (l + p_3)^2$, $a = 1$, $b = 1$ and $l = 3$
- $\mathcal{W} = \{(x, q) \in \mathbb{R}^2 : f(x, q, p) = 0 \wedge x \in [-3, 5] \wedge q \in [-1, 3]\}$
- A priori maximal perturbation: $\Delta = 0.3$

Parametric Kantorovich constants definition

$$\begin{aligned}
\chi &\geq \max_{\substack{(x,q)\in\mathcal{W}\\ \|p\|\leq\Delta}} \frac{1}{2\,|x - a - p_1|} \\
\delta &\geq \max_{\substack{(x,q)\in\mathcal{W}\\ \|p\|\leq\Delta}} \frac{|\,(x - a - p_1)^2 + (q - b - p_2)^2 - (l + p_3)^2\,|}{2\,|x - a - p_1|} \\
\gamma &\geq \max_{\substack{(x,q)\in\mathcal{W}\\ \|p\|\leq\Delta}} \frac{|x - a - p_1| \;+\; |q - b - p_2| \;+\; |l + p_3|}{|x - a - p_1|}\ .
\end{aligned}$$

f quadratic wrt x and $p \Rightarrow$ Lipschitz constants $\lambda = 2$ and $\mu = 2$

The PRRP robot

Parametric Kantorovich constants computation

- Nonlinear non-smooth problems
- IBEX[a] [CJ09, TANC11, ATNC14]: branch and bound + numerical constraint programming + linearization: $\chi = 0.26$, $\delta_0 = 1.1$ and $\gamma = 3.90$ (less than 0.01*s*)

[a] Available at `http://www.ibex-lib.org/`

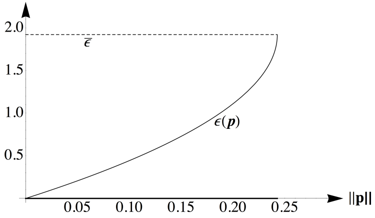
Bounds

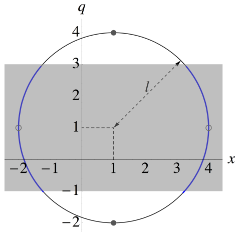
- $\delta_1(p) = 3.9p + 0.26p^2$
- $\mathcal{P} = \{p \in \mathbb{R}^3 : \|p\| \leq 0.24\}$
- $\overline{\epsilon} = 1.93$

- $\epsilon(p) = 1.93(1 - \sqrt{1 - 1.04\delta(p)})$

  $\epsilon(p) \approx \gamma\|p\|$ for $\|p\| \ll 1$

  (correct asymptotic is 3.82 $\rightarrow$ overestimated)





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