Analysis of acoustic-gravity waves in a free-surface compressible and stratified ocean model

Laurent Debreu, Emilie Duval, Eric Blayo, Inria, AIRSEA team

F. Auclair (UPS, Toulouse), P. Marchesiello (IRD, Toulouse)

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Plan

- Pourquoi s'intéresser au non-hydrostatique compressible ?
- Systèmes d'équations compressibles
- Analyse des ondes

Why trying to solve the non-hydrostatic compressible set of equations ?

Computational reasons: pressure solution

Non hydrostatic (Boussinesq) equations

$$\partial_x u + \partial_z w = 0$$

$$\partial_t u = -\partial_x p_h - \partial_x q$$

$$\partial_t w = -\left(\partial_z p_h + \partial_z q + \rho g\right)$$

$$\rho = \rho(T, S, p)$$

Pressure decomposition:

$$p = p_{\mathrm{H}} + q, \quad p_{\mathrm{H}} = \rho_0 g \eta + g \int_z^0 \rho(x, z', t) \mathrm{d}z'$$

Boundary conditions:

$$\partial_t \eta = w(0),\ w(-H) = 0,\ q(0) = 0$$

Pressure correction method requires the solution of:

$$\Delta q = F(u, \rho), \quad q \text{ non hydrostatic pressure}$$

According to the choice of the vertical coordinates system, the corresponding Poisson matrix can be ill-conditionned. Iterative methods are slow to converge.

→ (pseudo)-compressible approach

Computational reasons: barotropic/baroclinic coupling

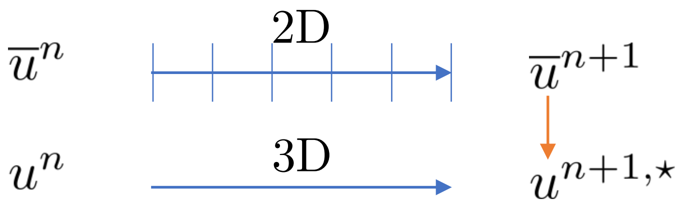
Ondes rapides (barotropes liées à la surface libre), $\sqrt{gH}$
Ratio autour de 60
-> séparation barotrope/barocline $u = \bar{u} + u'$

$$\overline{u} = \frac{1}{H+\eta}\int_{-H}^{n} u(x,z)\mathrm{d}z$$

Ondes lentes (baroclines, liées à la stratification) $\dfrac{NH}{n\pi}$

Split explicit approach

$\sqrt{gH}\delta t/\Delta x \leq 1$



$\delta t = \Delta_t/N$

Δt

In a non-hydrostatic incompressible approach, $u^{n+1,\star}$ has to be corrected both by the barotropic velocity and by the pressure correction

It rigorously works only when a surface homogeneous Neumann boundary condition is applied on non-hydrostatic pressure q

It prevents the accurate simulation of surface non-hydrostatic waves

Physically, we can expect that non-hydrostatic processes have to be integrated with a time step at most equal to the one of surface waves → Split-explicit approach

Solving the compressible set of equations

Pseudo-compressibility approach

Pseudo-compressible non hydrostatic equations

$$\frac{1}{c_s^2}\frac{\partial q}{\partial t} + \partial_x u + \partial_z w = 0$$

$$\partial_t u = -\partial_x p_h - \partial_x q$$

$$\partial_t w = -\left(\partial_z p_h + \partial_z q + \rho g\right)$$

$$\rho = \rho(T, S, p_h) + \frac{1}{c_s^2} q$$

In practice, c_s, the speed of sound, is chosen such that $\epsilon_a = \frac{\sqrt{gH}}{c_s} \ll 1$

Pseudo-compressibility and implicit schemes

Pseudo-compressible non hydrostatic equations

$$\frac{1}{c_s^2}\frac{\partial q}{\partial t} + \partial_x u + \partial_z w = 0$$

$$\partial_t u = -\partial_x p_h - \partial_x q$$

$$\partial_t w = -\left(\partial_z p_h + \partial_z q + \rho g\right)$$

$$\rho = \rho(T, S, p_h) + \frac{1}{c_s^2} q$$

$$\frac{1}{c_s^2}\frac{q^{n+1} - q^n}{\Delta t} = -\nabla \cdot U^{n+1}$$

$$\frac{U^{n+1} - U^n}{\Delta t} = -\nabla q^{n+1} + G$$

$$-\frac{q}{c_s^2 \Delta t^2} + \Delta q = F(u, \rho, q)$$

Bien meilleur conditionnement

Pseudo-compressibility approach

Pseudo-compressible non hydrostatic equations

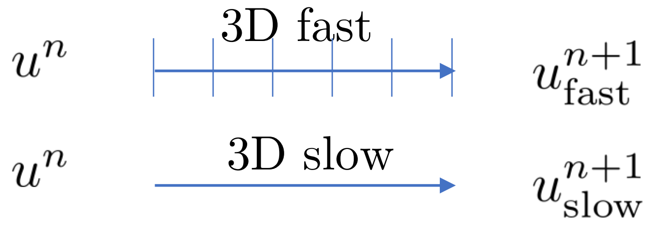
$$\frac{1}{c_s^2}\frac{\partial q}{\partial t} + \partial_x u + \partial_z w = 0$$

$$\partial_t u = -\partial_x p_h - \partial_x q$$

$$\partial_t w = -\left(\partial_z p_h + \partial_z q + \rho g\right)$$

$$\rho = \rho(T, S, p_h) + \frac{1}{c_s^2} q$$

$$c_s\,\delta t/\Delta x \leq 1, \quad \sqrt{gH}\delta t/\Delta x \leq \epsilon_a$$



The barotropic and acoustic waves are integrated with the same (small) time step

Use of the (Non Boussinesq) compressible solutions

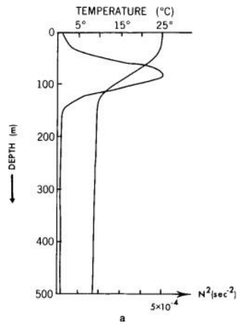
- Extreme atmospheric events
  e.g. Airburst generated tsunamis
- Steric effect
- Tsunamis detection
- Sound waves ?

→ Compressibility effects on linear waves (Auclair et al, 2020)

Linear waves in a free surface, stratified and compressible ocean

Linearized equations
and boundary conditions

$\hat{\rho}_h(z)$



$$\frac{\partial \hat{\rho}_h(z)}{\partial z} = -\frac{1}{D(z)}\,\hat{\rho}_h(z)$$

$$\frac{1}{D(z)} = \frac{N^2(z)}{g} + \frac{g}{c_s^2}$$

$$\frac{\partial \hat{p}_h(z)}{\partial z} = -\hat{\rho}_h(z) g$$

$$u = 0 + u', \rho = \hat{\rho}_h(z) + \rho', p = \hat{p}_h + p'$$

$$\begin{aligned}
\frac{\partial \hat{\rho}_h u}{\partial t} &= -\frac{\partial p}{\partial x} \\
\frac{\partial \hat{\rho}_h w}{\partial t} &= -\frac{\partial p}{\partial z} - \rho g \\
\frac{\partial \rho}{\partial t} &= -\left(\frac{\partial \hat{\rho}_h u}{\partial x} + \frac{\partial \hat{\rho}_h w}{\partial z}\right) \\
\frac{\partial p}{\partial t} &= -c_s^2\left(\frac{\partial \hat{\rho}_h u}{\partial x} + \frac{\partial \hat{\rho}_h w}{\partial z}\right) - \left(c_s^2 N^2/g\right)\hat{\rho}_h w
\end{aligned}$$

$$\frac{\mathrm{d}\rho}{\mathrm{dt}} = \frac{1}{c_s^2}\frac{\mathrm{d}p}{\mathrm{dt}}$$

with (flat) bottom and surface conditions:

$$\begin{aligned}
w(z=-H) &= \;=0 \\
\frac{\partial p}{\partial t}(z=0) &= \hat{\rho}_h\, g\, w(z=0)
\end{aligned}$$

Linear waves in a free surface, stratified and compressible ocean

$$\begin{pmatrix} \hat{\rho}_h u \\ \hat{\rho}_h w \\ \rho \\ p \end{pmatrix} = \begin{pmatrix} \widetilde{U}(z) \\ \widetilde{W}(z) \\ \widetilde{\rho}(z) \\ \widetilde{p}(z) \end{pmatrix} e^{i(k_x x - \Omega t)}$$

$$\downarrow$$

$$\widetilde{W}''(z) + \frac{1}{D(z)}\widetilde{W}'(z) + \left(k_x^2 \frac{N^2 - \Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{D'(z)}{D^2(z)}\right)\widetilde{W}(z) = 0$$

$$\widetilde{W}'(0) + \left(\tfrac{1}{D(0)} - \tfrac{g k_x^2}{\Omega^2}\right)\widetilde{W}(0) = 0$$

$$\widetilde{W}(-H) = 0$$

Linear waves in a free surface, stratified and compressible ocean

Limiting cases

Waves	Assumptions	Frequency (Ω)	Vertical wavenumber k_z
Acoustic gravity modes (unstratified, rigid lid)	Compressible, bounded	$\Omega_{am}^2 = c_s^2(k_x^2 + k_z^2)$	$k_{z,am} = \frac{\pi}{2H} + \frac{m\pi}{H}$
External gravity waves (unstratified, incompressible)	Free-surface	$\Omega_{sw}^2 = gk_x \tanh(k_x H)$	$k_{z,sw} \approx k_x$
Internal gravity modes (incompressible, rigid lid)	Stratified, bounded	$\Omega_{im}^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2}$	$k_{z,im} = \frac{n\pi}{H}$

Table 1: Simplified models of ocean waves and their dispersion relations in a vertical section for a bounded ocean (bottom). Ω is the time frequency of the wave, k_x and k_z are wavenumbers, g is the acceleration of gravity, H a reference depth, N a reference Brunt-Väisälä frequency and c_s the speed of sound. n and m are two positive integer numbers.

Constant Brunt Väisälä case

$$\widetilde{W}(z) = \widetilde{W}(0)\,\exp\left(-\frac{z}{2D_0}\right)\,F(z) \qquad \frac{1}{D_0} = \frac{N_0^2}{g} + \frac{g}{c_s^2}$$

$$F''(z) + \overbrace{\left(k_x^2\,\frac{N_0^2-\Omega^2}{\Omega^2} + \frac{\Omega^2}{c_s^2} - \frac{1}{4D_0^2}\right)}^{\equiv k_z^2} F(z) = 0$$

$$F'(0) + \left(\frac{1}{2D_0} - \frac{gk_x^2}{\Omega^2}\right) F(0) = 0$$

$$F(-H) = 0$$

$$F(z) = \frac{\sin\left(k_z(H+z)\right)}{\sin(k_z H)}$$

Profil vertical

- Oscillatoire si kz réel
- Exponential si kz imaginaire pur

Relations de dispersion

Stratification

Compressibilité

Nombres caractéristiques :

$$\epsilon_i^2 = \frac{N^2 H}{g}, \quad \epsilon_a^2 = \frac{gH}{c_s^2}$$

$$\omega = \Omega\sqrt{\frac{H}{g}}, \quad \delta_x = k_x H, \quad \delta_z = k_z H$$

$$\delta_x^2 + \delta_z^2 = \epsilon_i^2 \frac{\delta_x^2}{\omega^2} + \epsilon_a^2 \omega^2 - \frac{(\epsilon_a^2 + \epsilon_i^2)^2}{4}$$

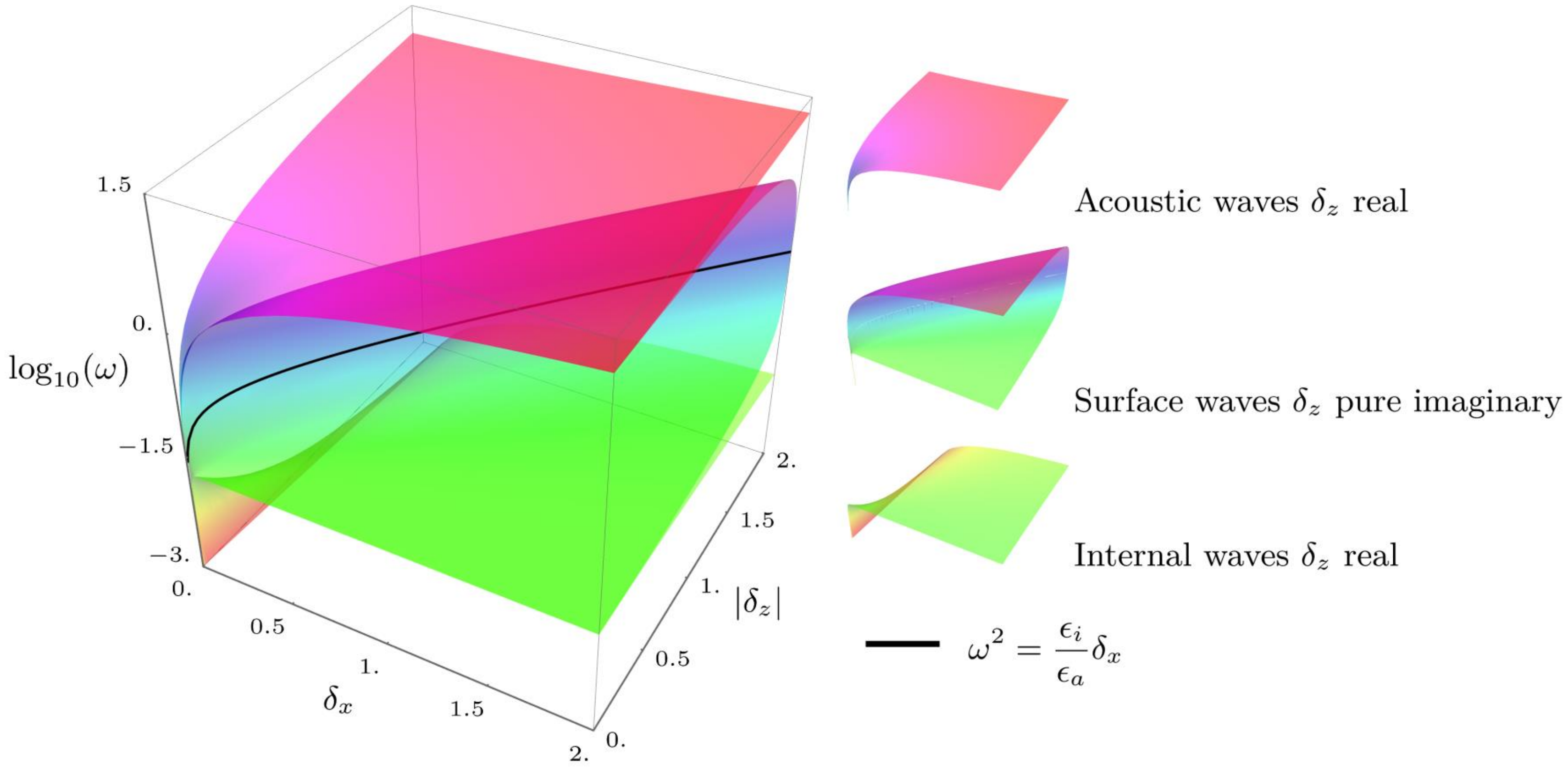
Inner dispersion relation

$$\omega^2 = \frac{\delta_x^2 \tan(\delta_z)}{\delta_z + \frac{\epsilon_a^2 + \epsilon_i^2}{2} \tan(\delta_z)}$$

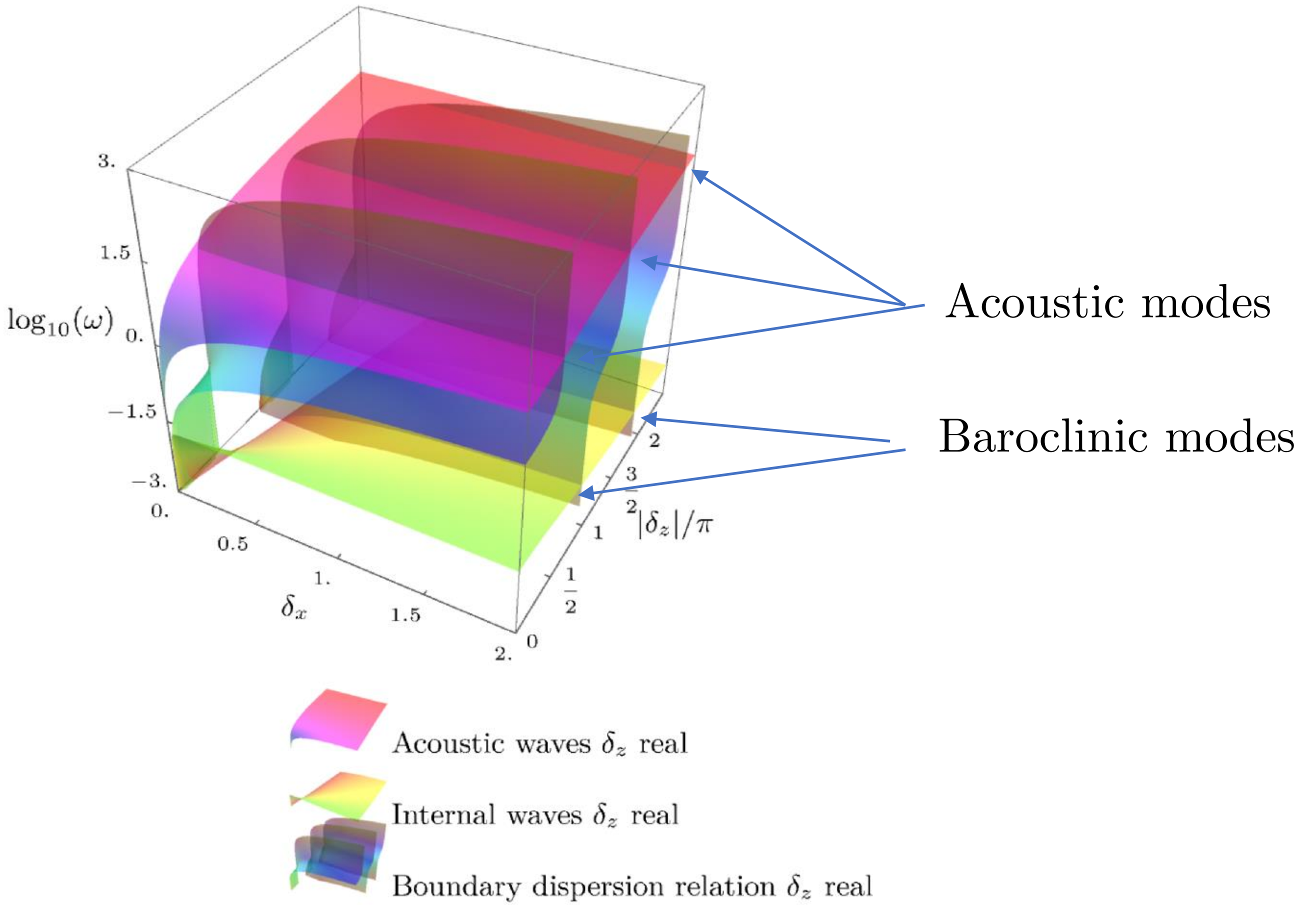
Boundary dispersion relation

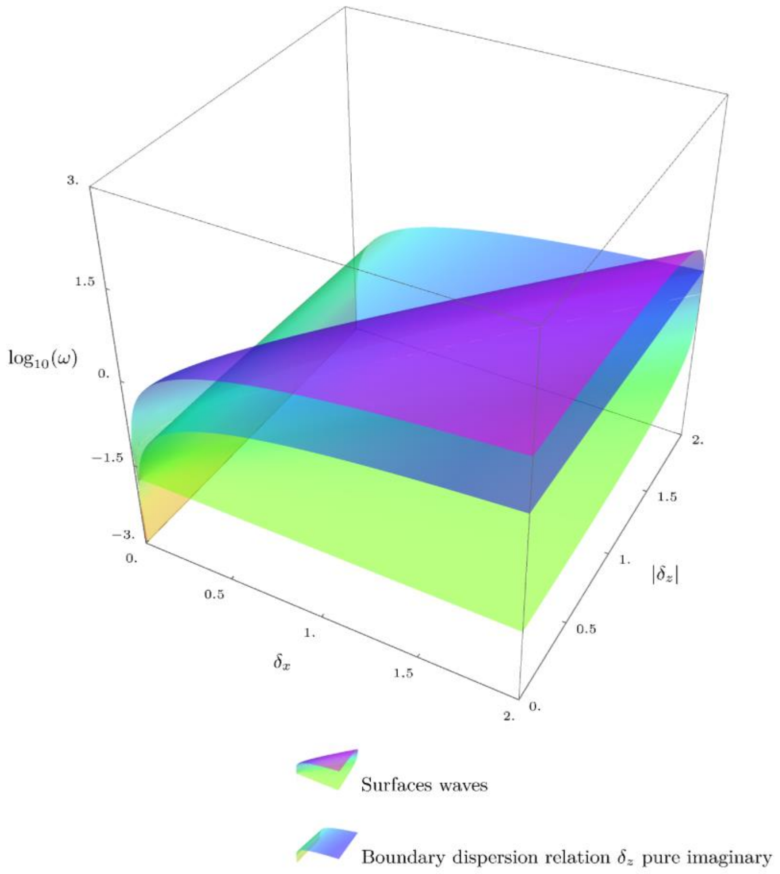
Si $\epsilon_a = 0$ (incompressible) ou si $\max(\epsilon_a, \epsilon_i) < \sqrt{2}$, alors ω est réel, (et δz est réel ou imaginaire pur).

Inner dispersion relation

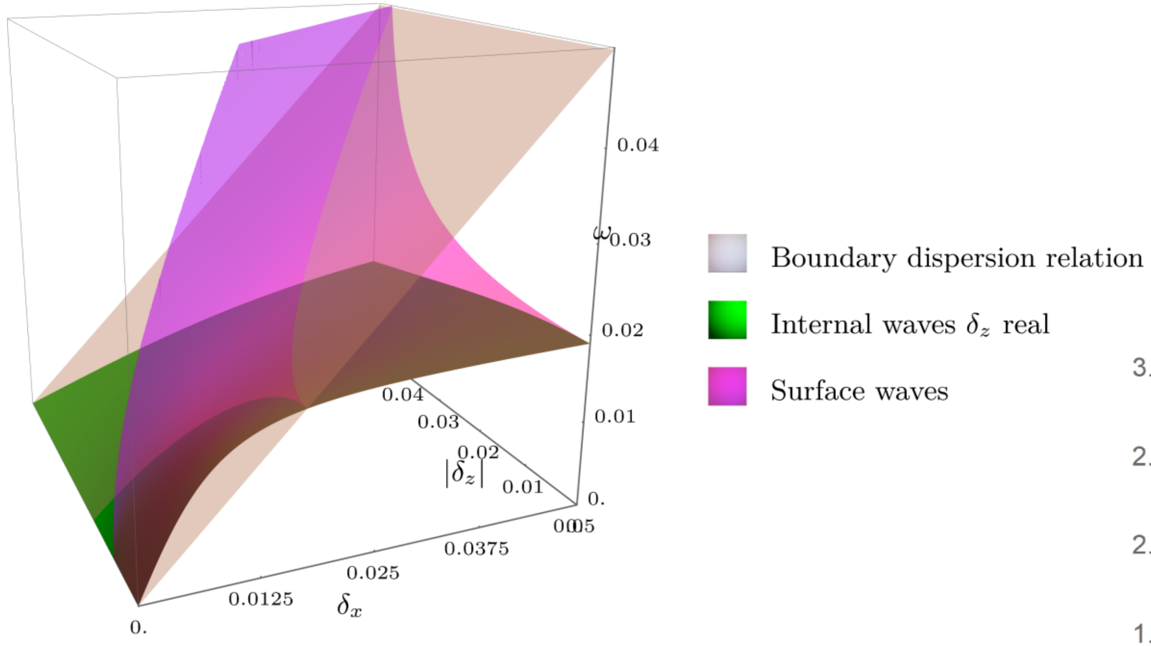


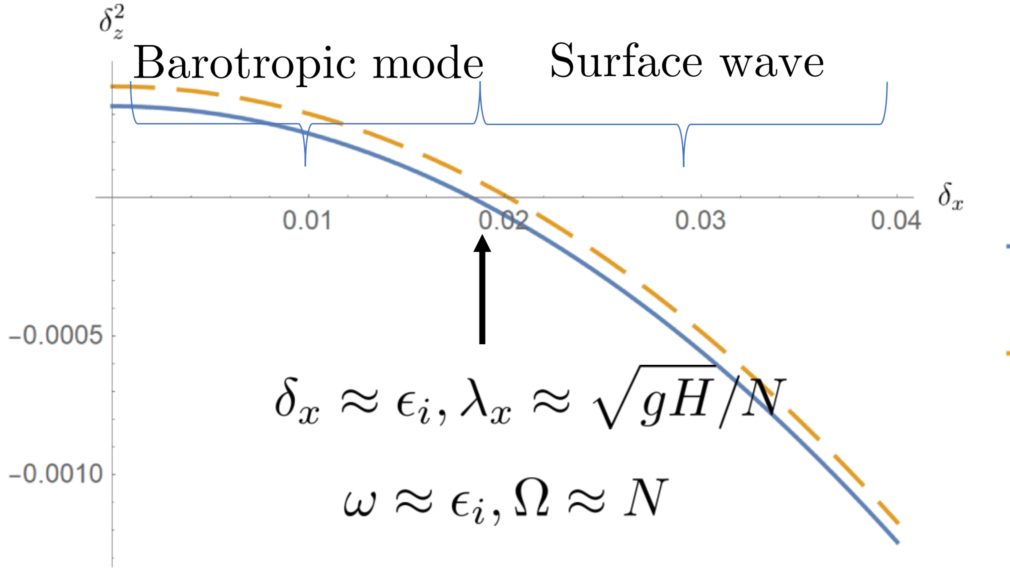
Inner and boundary dispersion relations

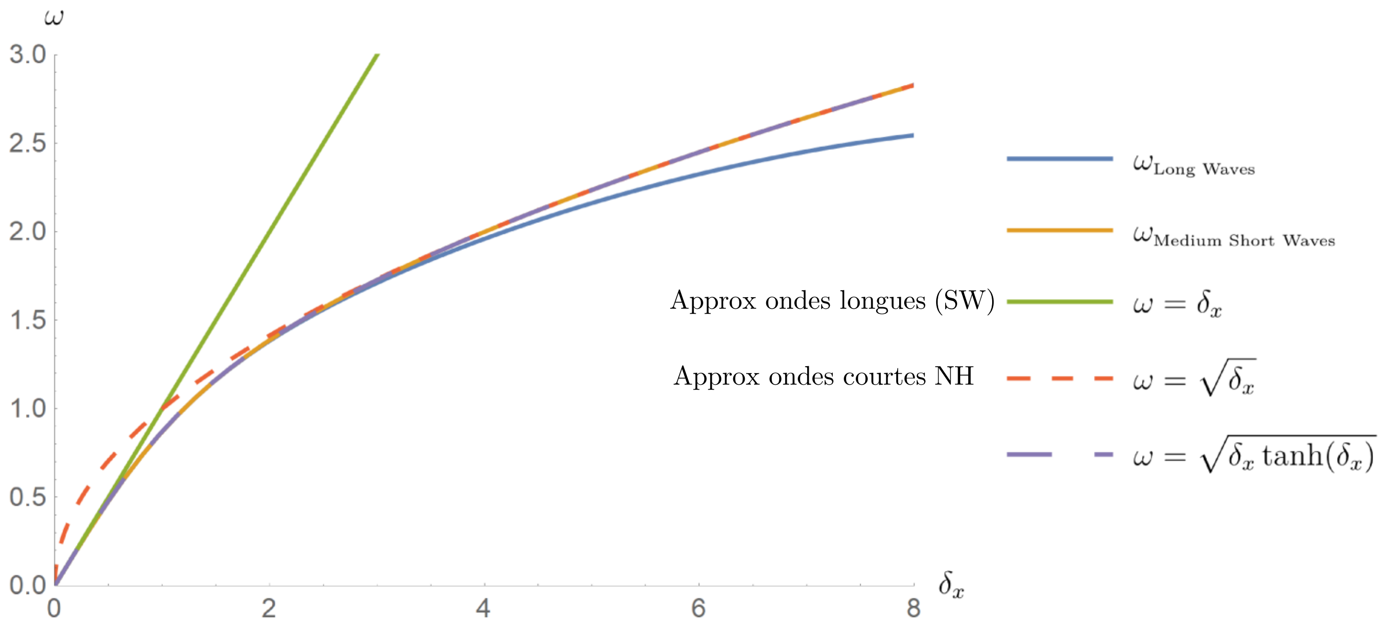




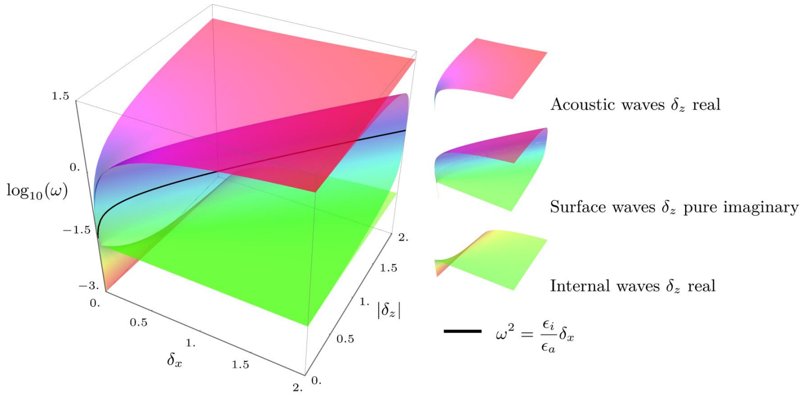
Long waves $|\delta z| \ll 1$







Linear waves in a free surface, stratified and compressible ocean



Waves	Frequency (Ω)	Vertical wavenumber k_z
Modified Acoustic Modes (MAM) $k_{z,m} = \frac{1}{H}(\pi/2 + m\pi), m \geq 0$	$\Omega^2_{mam} = c_s^2(k_x^2+k_z^2)\left[1-\frac{g}{Hc_s^2}\frac{k_x^2-k_{z,m}^2}{(k_x^2+k_{z,m}^2)^2}+\frac{N^2}{gH}\frac{1}{k_x^2+k_{z,m}^2}\right]$	$k_{z,m}\left[1-\frac{g}{Hc_s^2}\frac{k_x^2-k_{z,m}^2}{2(k_x^2+k_{z,m}^2)k_{z,m}^2}+\frac{N^2}{2gH^2}\frac{1}{k_{z,m}^3}\right]$
Modified Internal Modes (MIM) $k_{z,n} = \frac{1}{H}(n\pi), m \geq 1$	$\Omega^2_{mim} = \frac{N^2k_x^2}{k_x^2+k_{z,n}^2}\left[1-2\frac{N^2}{gH}\frac{k_{z,n}^2}{(k_x^2+k_{z,n}^2)^2}\right]$	$k_{z,n}\left[1+\frac{N^2}{gH}\frac{1}{k_x^2+k_{z,n}^2}+\left(\frac{N^2}{gH}\right)^3\frac{(k_x^2-k_{z,n}^2)}{(k_x^2+k_{z,n}^2)^3}\right]$
Barotropic mode $k_x \leq k_{x,0} \approx \frac{N^2}{g}$	$\Omega^2 = gHk_x^2\left[1-\frac{1}{6}\left(\epsilon_i^2+3\epsilon_a^2\right)\right]$	$\sqrt{k_{x,0}^2-k_x^2}$
Medium and short surface waves $k_x \geq k_{x,0} \approx \frac{N^2}{g}$	$\Omega^2 = gk_x\tanh(Hk_x)$	$ik_x\sqrt{1-\frac{1}{k_x}\left(\frac{N^2}{g\tanh(Hk_x)}+\frac{g}{c_s^2}\tanh(Hk_x)\right)}$

Table 1: Compressibility and stratification induced modifications to the usual dispersion relations. Ω is wave angular frequency, k_x and k_z are wavenumbers, g is the acceleration of gravity, H the water depth, N a reference Brunt-Väisälä frequency and c_s the speed of sound. $\epsilon_i^2 = \frac{N^2H}{g}, \epsilon_a^2 = \frac{gH}{c_s^2}$.

Solutions analytiques :

$$W(z) = e^{-z/(2D_0)}F(z) = e^{-z/2D_0}\frac{\sin(k_z(H+z))}{\sin(k_zH)}$$

$$P(z) = \frac{\Omega^2}{g^2k_x^2(1-(\Omega/(c_sk_x))^2)}\left(N_0^2W(z)+gW'(z)\right)$$

$$\eta(x,t) = \eta_0\cos(k_xx)\cos(\Omega t)$$

$$u(x,z,t) = \eta_0\frac{gk_x}{\Omega}\sin(k_xx)\sin(\Omega t)\frac{\hat{\rho}_h(0)}{\hat{\rho}_h(z)}P(z)$$

$$w(x,z,t) = -\eta_0\Omega\cos(k_xx)\sin(\Omega t)\frac{\hat{\rho}_h(0)}{\hat{\rho}_h(z)}W(z)$$

$$p(x,z,t) = \eta_0g\hat{\rho}_h(0)\cos(k_xx)\cos(\Omega t)P(z)$$

$$\rho(x,z,t) = \eta_0g\hat{\rho}_h(0)\cos(k_xx)\cos(\Omega t)\left(\frac{N_0^2}{g^2}W(z)+\frac{1}{c_s^2}P(z)\right)$$

Interactions potentielles dans le cadre du défi SURF

- Autres techniques de pseudo-compressibilité
- Modélisation des ondes sonores (modélisation directe et problèmes inverses (quel modèle pour quel problème inverse ?)
- Conditions aux limites couplage nonhydrostatique / hydrostatique
- Etude des familles d'ondes dans un cadre plus complexe