Numerical methods for free surface Navier-Stokes equations

Variable density flows and dispersive waves

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also with

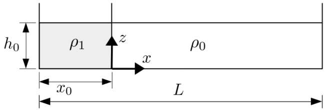
- A-S. Bonnet Ben-Dhia, M.-O. Bristeau, J. Dubois
- E. Godlewski, S. Imperiale, A. Mangeney, S. Téchéné

STUOD conference - September 2020

Outline

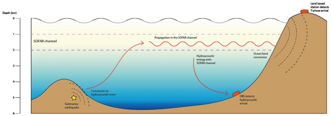
Importance of numerical analysis and stable schemes

- lock-exchange phenomenon
- comparison of several numerical codes



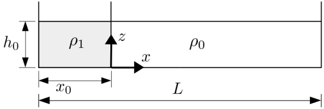
Pseudo-compessible models

- allow to capture acoustic waves
- efficient resolution of dispersive terms
  - ○ presentation a simplified context to single out the main ideas

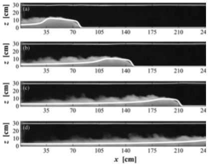


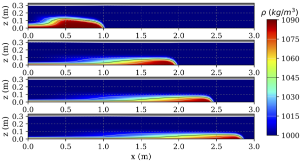
A typical example

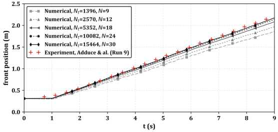
- The experimental device [Adduce *et al.*, J. Hydraul. Eng. 2012] (anim) (anim_low)



- Measurements versus simulation results

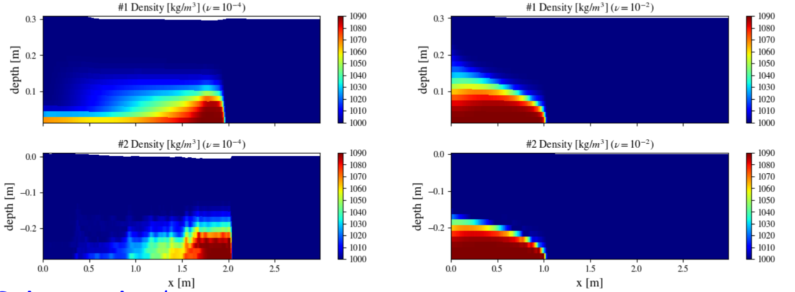




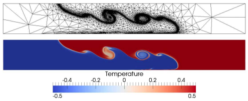


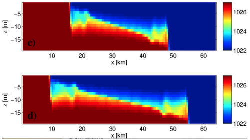
Importance of num. analysis & schemes

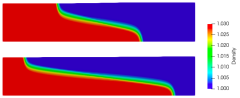
Comparison of 2 different codes (one of the experiments proposed by Adduce 2012)



Other codes/test case







See also Wroniszewski et al. Benchmarking of Navier-Stokes codes for free surface simulations, Coastal Eng., 91:1-17,2014

Outline

Incompressible & compressible Euler system (+ B.C.)

$$\begin{cases} \nabla.\underline{\mathbf{u}} = 0 \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = \mathbf{G} \end{cases} \quad \Bigg| \quad \begin{cases} \dot{p} + \nabla.(\underline{\mathbf{u}}p) + \rho_0 c^2 \nabla.\underline{\mathbf{u}} = 0 \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = \mathbf{G} \end{cases}$$

$c \gg 1$ is the sound speed

Discrete scheme

- Projection-correction

$$\begin{cases} \nabla.\underline{\mathbf{u}}^{n+1} = 0 \\ \underline{\mathbf{u}}^{n+1} = \underline{\mathbf{u}}^{n+1/2} - \Delta t^n \nabla p^{n+1} \end{cases}$$

- An elliptic equation

$$\Delta p^{n+1} = \frac{1}{\Delta t^n} \nabla \cdot \mathbf{u}^{n+1/2}$$

- Explicit scheme

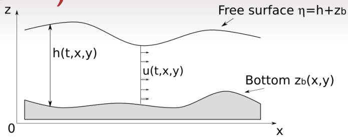
$$\begin{cases} p^{n+1} = p^{n*} - \Delta t^n \rho_0 c^2 \nabla \cdot \underline{\mathbf{u}}^{n*} \\ \underline{\mathbf{u}}^{n+1} = \underline{\mathbf{u}}^{n*} - \Delta t^n \nabla p^{n*} \end{cases}$$

- A wave-type equation

$$p^{n*,k+1} = 2p^{n*,k} - p^{n*,k-1} + \frac{(\Delta t^n \rho_0 c^2)^2}{K^2} \Delta p^{n*,k}$$

Many references in the literature

A SW dispersive model (Green-Naghdi...)



$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0$$
$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{g}{2}h^2 + hp\right) = -(gh + \frac{\gamma^2}{2}p)\frac{\partial z_b}{\partial x}$$
$$\frac{\partial (hw)}{\partial t} + \frac{\partial (huw)}{\partial x} = \gamma p$$
$$\gamma w + h\frac{\partial u}{\partial x} - \frac{\gamma^2}{2}u\frac{\partial z_b}{\partial x} = 0$$

Aissiouene et al.
J. Comp. Math. 2020

In a more compact form (in 2d) $\mathbf{u} = (u, v, w)^T$

$$\begin{aligned} &\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0 \\ &\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0(\frac{g}{2}h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b) \\ &\mathrm{div}_{sw}(\mathbf{u}) = 0 \end{aligned} \quad (1)$$

with

$$\nabla_{sw} f = \begin{pmatrix} h\frac{\partial f}{\partial x} + f\frac{\partial \zeta}{\partial x} \\ h\frac{\partial f}{\partial y} + f\frac{\partial \zeta}{\partial y} \\ -\gamma f \end{pmatrix}, \quad \mathrm{div}_{sw}(\mathbf{w}) = \gamma w_3 + h\frac{\partial w_1}{\partial x} + h\frac{\partial w_2}{\partial y} - w_1\frac{\gamma^2}{2}\frac{\partial z_b}{\partial x} - w_2\frac{\gamma^2}{2}\frac{\partial z_b}{\partial y}$$

Idea of the numerical scheme

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0$$
$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0(\frac{g}{2}h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b)$$
$$\mathrm{div}_{sw}(\mathbf{u}) = 0$$

- Hyperbolic step (shallow water equations)

$$h_i^{n+1/2} = h_i^n - \sigma_i\big(\mathcal{F}_{h,i+1/2} - \mathcal{F}_{h,i-1/2}\big)$$
$$(h\mathbf{u})_i^{n+1/2} = (h\mathbf{u})_i^n - \sigma_i\big(\mathcal{F}_{h\mathbf{u},i+1/2} - \mathcal{F}_{h\mathbf{u},i-1/2}\big)$$

- Correction step

$$h_i^{n+1} = h_i^{n+1/2}$$
$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^{n+1/2} - \frac{\Delta t^n}{h_i^{n+1}} \nabla_{sw}\left(p^{n+1}\right)_i \quad (2)$$
$$\mathrm{div}_{sw}\left(\mathbf{u}^{n+1}\right)_i = 0 \quad (3)$$

implying to solve (very costly $\approx \mathcal{O}(N^{3/2})$)

$$\mathrm{div}_{sw}\left(\frac{1}{h^{n+1}}\nabla_{sw}\left(p^{n+1}\right)\right)_i = \frac{1}{\Delta t^n}\mathrm{div}_{sw}\left(\mathbf{u}^{n+1/2}\right)_i$$

Now we start from the compressible NS

The compressible Navier-Stokes-Fourier system

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \\
&\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p - \nabla \cdot \sigma = \rho \mathbf{g} \\
&\frac{\partial}{\partial t}\left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e\right) + \nabla \cdot \left(\left(\rho \frac{|\mathbf{U}|^2}{2} + \rho e + p - \sigma\right)\mathbf{U}\right) = -\lambda \nabla T + \rho \mathbf{g} \cdot \mathbf{U}
\end{aligned}
$$

Using thermodynamical relations (without diffusion/viscosity)

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \\
&\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g} \\
&\frac{\partial (\rho p)}{\partial t} + \nabla \cdot (\rho \mathbf{U} p) + \rho^2 c^2 \nabla \cdot \mathbf{U} = 0
\end{aligned} \tag{4}
$$

where $c \gg 1$ is the sound speed

Derivation of the model (Galerkin approx.)

Depth averaging of the compressible Euler system

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta}\rho dz + \nabla_{x,y}\cdot\int_{z_b}^{\eta}\rho\mathbf{v} dz = 0$$
$$\frac{\partial}{\partial t}\int_{z_b}^{\eta}\rho\mathbf{v} dz + \nabla_{x,y}\cdot\int_{z_b}^{\eta}\rho\mathbf{v}\otimes\mathbf{v} dz + \nabla_{x,y}\int_{z_b}^{\eta} p dz = p(t,\mathbf{x},z_b(\mathbf{x}))\nabla_{x,y} z_b$$
$$\frac{\partial}{\partial t}\int_{z_b}^{\eta}\rho u_3 dz + \nabla_{x,y}\cdot\int_{z_b}^{\eta}\rho u_3\mathbf{v} dz = p(t,\mathbf{x},z_b(\mathbf{x})) - \int_{z_b}^{\eta}\rho g dz$$
$$\frac{\partial}{\partial t}\int_{z_b}^{\eta}\rho p dz + \nabla_{x,y}\cdot\int_{z_b}^{\eta}\rho p\mathbf{v} dz + \rho_0^2 c^2\int_{z_b}^{\eta}\nabla\cdot\mathbf{U} dz = 0$$

Choice of the approximation space $U=(u_1,u_2,u_3)^T$, $\mathbf{v}=(u_1,u_2)^T$

$$u_1(t,\mathbf{x},z) = u(t,\mathbf{x}),\quad u_2(t,\mathbf{x},z) = v(t,\mathbf{x}),\quad \rho(t,\mathbf{x},z) = \rho(t,\mathbf{x})$$
$$u_3(t,\mathbf{x},z) = \varphi_\delta\left(\frac{\eta - z}{h}\right) w(t,\mathbf{x})$$
$$p(t,\mathbf{x},z) = \rho g(\eta - z) + \psi_\delta\left(\frac{\eta - z}{h}\right) P(t,\mathbf{x})$$

φ_δ and ψ_δ two given functions with few constraints (linear along z)

The obtained model

A shallow water compressible Euler system

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0 (\frac{g}{2} h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b)$$

$$\frac{\partial}{\partial t}(hp + \frac{gh^2}{2}) + \nabla_0 \cdot \Big((hp + \frac{gh^2}{2})\mathbf{u} \Big) + c^2 \mathrm{div}_{sw}\,(\mathbf{u}) = 0$$

A shallow water dispersive Euler system

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0 (\frac{g}{2} h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b)$$

$$\mathrm{div}_{sw}\,(\mathbf{u}) = 0$$

The shallow water system

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0 (\frac{g}{2} h^2) = -gh\nabla_0(z_b)$$

Dispersion relation

Gravity waves and acoustic waves

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0 (\frac{g}{2} h^2) + \nabla_{sw}\, p = -gh\nabla_0 (z_b)$$

$$\frac{\partial}{\partial t}(hp + \frac{gh^2}{2}) + \nabla_0 \cdot \left((hp + \frac{gh^2}{2})\mathbf{u} \right) + c^2 \mathrm{div}_{sw}\, (\mathbf{u}) = 0$$

- Linearized version (acoustic)

$$\frac{\partial^2 p}{\partial t^2} - c^2 \mathrm{div}_{sw} \cdot \nabla_{sw}\, p = S$$

- Dispersion relation

$$\frac{\omega}{k} = u \pm \gamma \sqrt{\frac{gh + p}{\gamma^2 - (hk)^2}} + \mathcal{O}\left(\frac{1}{c^2}\right), u \pm c \sqrt{1 + \frac{\gamma^2}{(hk)^2}} + \mathcal{O}\left(\frac{1}{c}\right)$$

- Surface & acoustic waves ... but only in a shallow water context

Semi-discrete (in time) scheme ($c^2 = 1/\varepsilon$)

The model to approximate

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0 \qquad (5)$$

$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0 (\frac{g}{2} h^2) + \nabla_{sw}\, p = -gh\nabla_0 (z_b) \qquad (6)$$

$$\frac{\partial}{\partial t}(hp + \frac{gh^2}{2}) + \nabla_0 \cdot \left((hp + \frac{gh^2}{2})\mathbf{u} \right) + \frac{1}{\varepsilon}\mathrm{div}_{sw}\,(\mathbf{u}) = 0 \qquad (7)$$

The time scheme ($X = (h, h\mathbf{u})^T$)

The hyperbolic part (FV scheme constrained by a CFL condition)

$$\begin{cases} X^{n+1/2} = X^n - \Delta t^n \nabla_{x,y} \cdot F(X^n) + \Delta t^n S(X^n) \\ (hp)^{n+1/2} = (hp)^n - \frac{g}{2}\left((h^{n+1/2})^2 - (h^n)^2 \right) - \Delta t^n \nabla_0 \cdot \left(h^n (p^n + \frac{g}{2}(h^n)^2 \right. \end{cases}$$

The dispersive/acoustic part (explicit resolution of a wave equation with sul

$$\begin{cases} \mathbf{u}^{n+1/2,k+1} = \mathbf{u}^{n+1/2,k} - \frac{\Delta t^n}{K h^{n+1}} \nabla_{sw}\, p^{n+1/2,k+1} \\ p^{n+1/2,k+1} = p^{n+1/2,k} - \frac{\Delta t^n}{\varepsilon K h^{n+1}} \mathrm{div}_{sw}\, \mathbf{u}^{n+1/2,k} \end{cases} \quad k = 1, \dots, K$$

in other words

$$p^{n+1/2,k+1} = 2p^{n+1/2,k} - p^{n+1/2,k-1} + \frac{(\Delta t^n)^2}{\varepsilon K^2 h^{n+1}} \mathrm{div}_{sw} \left(\frac{1}{h^{n+1}} \nabla_{sw}\, p^{n+1/2,k} \right)$$

Idea of the numerical scheme $c^2 = 1/\varepsilon$

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0 \quad (8)$$

$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0(\frac{g}{2}h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b) \quad (9)$$

$$\frac{\partial}{\partial t}\left(hp + \frac{g}{2}h^2\right) + \nabla_0 \cdot \left(\mathbf{u}\left(hp + \frac{g}{2}h^2\right)\right) + \frac{1}{\varepsilon}\mathrm{div}_{sw}\,\mathbf{u} = 0 \quad (10)$$

- Hyperbolic step with a CFL constraint ($P = p + gh/2$)

$$\begin{cases} h_i^{n+1/2} = h_i^n - \sigma_i\big(\mathcal{F}_{h,i+1/2} - \mathcal{F}_{h,i-1/2}\big) \\ (h\mathbf{u})_i^{n+1/2} = (h\mathbf{u})_i^n - \sigma_i\big(\mathcal{F}_{h\mathbf{u},i+1/2} - \mathcal{F}_{h\mathbf{u},i-1/2}\big) \\ (hP)_{i+1/2}^{n+1/2} = (hP)_{i-1/2}^n - \Delta t^n \sigma_i\big(\mathcal{F}_{hP,i+1} - \mathcal{F}_{hP,i}\big) \end{cases}$$

- Correction step (sub-iterations)

$$\begin{cases} p^{n+1/2,k+1} = p^{n+1/2,k} - \frac{\Delta t^n}{\varepsilon K h^{n+1}}\mathrm{div}_{sw}\,\mathbf{u}^{n+1/2,k} \\ \mathbf{u}^{n+1/2,k+1} = \mathbf{u}^{n+1/2,k} - \frac{\Delta t^n}{K h^{n+1}}\nabla_{sw}\, p^{n+1/2,k+1} \end{cases}$$

or equivalently $\frac{\partial^2 p}{\partial t^2} - \frac{1}{\varepsilon}\mathrm{div}_{sw} \cdot \nabla_{sw}\, p = 0$

$$p^{n+1/2,k+1} = 2p^{n+1/2,k} - p^{n+1/2,k-1} + \frac{(\Delta t^n)^2}{\varepsilon K^2 h^{n+1}}\mathrm{div}_{sw}\left(\frac{1}{h^{n+1}}\nabla_{sw}\, p^{n+1/2,k}\right)$$

Properties of the scheme

$$
\begin{aligned}
&\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0 \\
&\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0(\tfrac{g}{2} h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b) \\
&\frac{\partial}{\partial t}\left(hp + \tfrac{g}{2} h^2\right) + \nabla_0 \cdot \left(\mathbf{u}\left(hp + \tfrac{g}{2} h^2\right)\right) + c^2 \mathrm{div}_{sw}\, \mathbf{u} = 0
\end{aligned}
$$

- Well-balancing (lake at rest)
- Num. treatment of the boundary conditions
- Domain invariant ($h_i^{n+1} \geq 0$)
- Discrete entropy (if a discrete entropy holds for SW)

Computational costs

Continuous models

$$\frac{\partial h}{\partial t} + \nabla_0 \cdot (h\mathbf{u}) = 0$$
$$\frac{\partial (h\mathbf{u})}{\partial t} + \nabla_0 \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla_0(\tfrac{g}{2} h^2) + \nabla_{sw}\, p = -gh\nabla_0(z_b)$$

$$\mathrm{div}_{sw}\, \mathbf{u} = 0$$

$$\frac{\partial h\left(p + \frac{g}{2} h^2\right)}{\partial t} + \nabla_0 \cdot \left(\mathbf{u} h\left(p + \frac{g}{2} h^2\right)\right) + \frac{1}{\varepsilon}\mathrm{div}_{sw}\, \mathbf{u} = 0$$

Discrete models

- Hyperbolic part $\mathcal{O}(N)$
- Elliptic step $\approx \mathcal{O}(N^{3/2})$

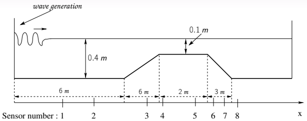
$$\mathrm{div}_{sw}\left(\frac{\nabla_{sw}\,(p_{nh}^{n+1})}{h^{n+1}}\right)_i = \frac{1}{\Delta t^n}\mathrm{div}_{sw}\,(\mathbf{u}^{n+1})_i$$

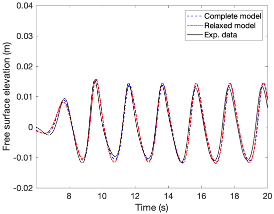
- Hyperbolic part $\mathcal{O}(N)$
- Wave equation $\mathcal{O}(KN) = \mathcal{O}\left(\frac{N}{\sqrt{\varepsilon}}\right) = \mathcal{O}\left(cN\right)$

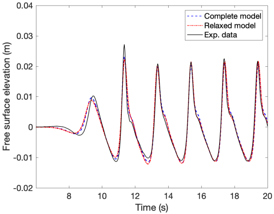
$$p^{n+1/2,k+1} = 2p^{n+1/2,k} - p^{n+1/2,k-1} + \frac{(\Delta t^n)^2}{\varepsilon K^2 h^{n+1}}\mathrm{div}_{sw}\left(\frac{\nabla_{sw}\, p^{n+1/2,k}}{h^{n+1}}\right)$$

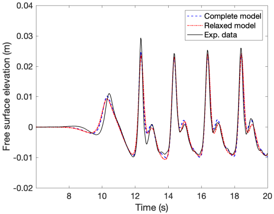
Numerical simulations

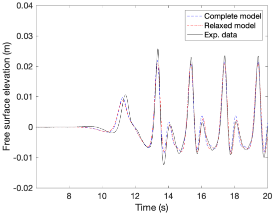
- Dinguemans experiments
- Animation
- Comparisons





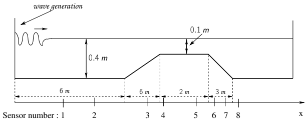


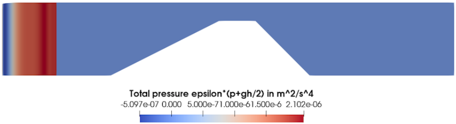


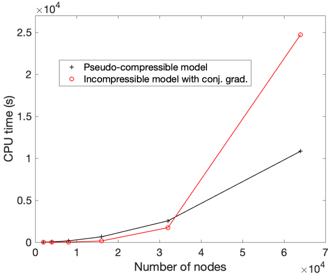


Numerical simulations (cont'd)

- Dinguemans experiments
- Animation
- Comparisons







Conclusion

- SW non-hydrostatic flows
  - models exist, numerical analysis : complex, non-intuitive behaviors
  - sensitivity to initial conditions, topography...
  - computational costs
- Acoustic waves
  - measured, rather linear propagation (SOFAR channel, density gradient, reflections)
  - mainly decoupled from gravity waves
  - early warning systems
- ◇ Interest of weakly compressible models
  - few works considering both acoustic and water waves
  - a physical phenomenon and a relaxation
  - pseudo-compressibility allows explicit schemes & reduced numerical costs
  - to be done in 3d (in progress)
- ◇ Only deterministic until now

The obtained model

A "1d" compressible Euler system

$$
\begin{aligned}
&\frac{\partial \eta}{\partial t} + u\frac{\partial(\eta+z_b)}{\partial x} = \gamma w\\
&\frac{\partial(\rho h)}{\partial t} + \nabla_0 \cdot (\rho h \mathbf{u}) = 0\\
&\frac{\partial(\rho h \mathbf{u})}{\partial t} + \nabla_0 \cdot (\rho h \mathbf{u} \otimes \mathbf{u}) + \nabla_0\left(\frac{\rho g}{2} h^2\right) + \nabla_{sw}\, p = -\rho g h \nabla_0(z_b)\\
&\frac{\partial}{\partial t}\left(\rho h p + \frac{\rho g}{2} h^2\right) + \nabla_0 \cdot \left(\mathbf{u}\left(\rho h p + \frac{\rho g}{2} h^2\right)\right) + \rho^2 c^2 \mathrm{div}_{sw}\, \mathbf{u} = 0
\end{aligned}
$$

completed with the energy balance

$$\frac{\partial}{\partial t}\Big(\rho h E\Big) + \frac{\partial}{\partial x}\Big(u(\rho h E + hp)\Big) = 0$$

with $E = \frac{u^2+w^2}{2} + g\frac{(\eta+z_b)}{2} + \frac{1}{2}\left(2 - \frac{\gamma^2}{2}\right) g z_b + \frac{p^2}{2c^2\rho}$