Stochastic modeling and hydrostatic balance

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Motivations

- Strong interest on the use of stochastic filters and ensemble methods for data assimilation and forecasting

⇒ Require stochastic version of the evolution law and/or a modeling of the dynamics errors

Several frameworks proposed in the literature (Berner et al 2017, Franzke et al. 2015)

- additive/multiplicative forcing (Buiza et al 99), backscatering (Leight 71), (Mason and Thomson 92)
- Low/fast modes decomposition (Majda et al. 99, Franzke et al 05), scale separation (Grooms and Majda).
- Approaches based on stochastic transport (Holm 15, Mémin 14)

Location Uncertainty (LU) Principles

- Stochastic flow :

$$\mathrm{d}\boldsymbol{X}_t = \underbrace{\boldsymbol{u}(\boldsymbol{X}_t, t)\mathrm{d}t}_{\text{large–scale / resolved}} + \underbrace{\boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t}_{\text{small–scale / unresolved}}$$

- Functional process :

$$\boldsymbol{\sigma}(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{B}_t = \int_{\Omega} \breve{\boldsymbol{\sigma}}(\boldsymbol{x}, \boldsymbol{y}, t)\mathrm{d}\boldsymbol{B}_t(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

$\boldsymbol{B}_t$ functional cyclindrical Wiener process and $\breve{\boldsymbol{\sigma}}$ is a bounded deterministic symmetric kernel $\Rightarrow$ $\boldsymbol{\sigma}$ Hilbert-Schmidt

- Spectral decomposition

$$\boldsymbol{\sigma}(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{B}_t = \sum_{n \in \mathbb{N}} \boldsymbol{\Phi}_n(\boldsymbol{x}, t)\mathrm{d}\beta_t^n$$

Location Uncertainty (LU) Principles

- Covariance operator :

$$Q(\boldsymbol{x},\boldsymbol{y},t,s) = \mathbb{E}\Big[\boldsymbol{\sigma}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{B}_t\big(\boldsymbol{\sigma}(\boldsymbol{y},s)\mathrm{d}\boldsymbol{B}_s\big)^T\Big]$$
$$= \delta(t-s)\mathrm{d}t\int_\Omega \breve{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{z},t)\breve{\boldsymbol{\sigma}}^T(\boldsymbol{y},\boldsymbol{z},s)\mathrm{d}\boldsymbol{z}$$

- Variance tensor (per unit of time) :

$$\boldsymbol{a}(\boldsymbol{x},t)\mathrm{d}t = Q(\boldsymbol{x},t)$$

- Turbulent Kinetic Energy (TKE) :

$$\mathrm{TKE} = \frac{1}{2}\frac{\mathrm{tr}(\boldsymbol{a})}{\mathrm{d}t} \qquad (m^2 \cdot s^{-2})$$

Stochastic Reynolds Transport Theorem (SRTT)

We assume that $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$ in the following.

- Rate of change of a scalar process θ within a volume transported by the stochastic flow :

$$\mathrm{d}\int_{\mathcal{V}(t)} \theta(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} = \int_{\mathcal{V}(t)} (\mathrm{D}_t\theta + \theta\,\boldsymbol{\nabla}\cdot\boldsymbol{u}^\star)\mathrm{d}\boldsymbol{x}$$

- Stochastic transport operator :

$$\mathrm{D}_t\theta \triangleq \mathrm{d}_t\theta + \underbrace{(\boldsymbol{u} - \frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{a})}_{\boldsymbol{u}^* = \boldsymbol{u} - \boldsymbol{u_s}} \cdot\boldsymbol{\nabla}\,\theta\mathrm{d}t + \underbrace{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\,\theta}_{\text{multiplicative noise}} - \underbrace{\boldsymbol{\nabla}\cdot(\frac{\boldsymbol{a}}{2}\boldsymbol{\nabla}\theta)\mathrm{d}t}_{\text{subgrid diffusion}}$$

$\boldsymbol{u} - \boldsymbol{u_s}$: corrected drift – effect of statistical inhomogeneity of the small-scale flow component; generalization of the Stokes drift; transitional regime (buffer zone in boundary layer turbulence)

[Bauer, Chandramouli, Chapron, Li & Mémin 2019a] [Pinier, Mémin, Laizet, & Lewandowski 2019]

Conservation Laws

For incompressible flow (with $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}^\star = 0$)

- Conservation of passive scalar :

$$\mathrm{D}_t\theta = 0$$

- Conservation of tracer energy : *[Resseguier, Memin & Chapron, 2017a]*

$$\mathrm{d}\int_\Omega \frac{1}{2}\theta^2 = \int_\Omega \theta \mathrm{d}_t\theta + \frac{1}{2}\mathrm{d}\langle\theta,\theta\rangle_t = 0$$

- Energy decomposition in terms of mean and variance fields :

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}(\mathbb{E}[\theta])^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\mathrm{Var}[\theta]$$

Stochastic Navier-Stokes Equations (SNS)

- Momentum equation:

$$\mathrm{D}_t \boldsymbol{u} + \boldsymbol{f} \times (\boldsymbol{u}\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t) = \boldsymbol{g}\mathrm{d}t - \frac{1}{\rho}\boldsymbol{\nabla}(p\mathrm{d}t + \mathrm{d}p_t^\sigma) + \nu\nabla^2(\boldsymbol{u}\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t)$$

- Continuity equation:

$$\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{u} - \boldsymbol{u}_s) = 0, \qquad \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma} = 0$$

- Mass equation:

$$\mathrm{D}_t \rho = 0$$

Stochastic Simple Boussinesq Equations (SSBE)

Weak compressibility: $\rho(\boldsymbol{x},t) = \rho_0 + \delta\rho(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t) = p_0(z) + \delta p(\boldsymbol{x},t)$
Hydrostatic balance: $\frac{\partial p_0}{\partial z} = -g\rho_0$

- Horizontal momentum equation:

$$\mathrm{D}_t \boldsymbol{u}^{\boldsymbol{h}} + f\boldsymbol{k} \times (\boldsymbol{u}^{\boldsymbol{h}}\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^{\boldsymbol{h}}) = b\boldsymbol{k} - \boldsymbol{\nabla}_{\boldsymbol{h}}(\phi\mathrm{d}t + \mathrm{d}\phi_t^\sigma)$$

- Vertical momentum:

$$\mathrm{D}_t w = -\frac{\partial}{\partial z}(\phi\mathrm{d}t + \mathrm{d}\phi_t^\sigma) + b\mathrm{d}t$$

- Continuity equation:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{u} - \boldsymbol{u}_s) = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$$

- Buoyancy equation:

$$\mathrm{D}_t(b + N^2 z) = 0$$

$\phi = \delta p/\rho_0$ and $\mathrm{d}\phi_t^\sigma = \mathrm{d}p_t^\sigma/\rho_0$: rescaled pressure fluct., $b = -g\delta\rho/\rho_0$: buoyancy

Stochastic Simple Boussinesq Equations (SSBE)

Adimentionnal buoyancy equation:

$$(\mathrm{d}_t b + (\boldsymbol{u} - \frac{1}{\Upsilon}\boldsymbol{u_s}) \cdot\boldsymbol{\nabla}\, b\mathrm{d}t - \frac{1}{2\Upsilon}\, \boldsymbol{\nabla}\cdot (\boldsymbol{a}\boldsymbol{\nabla} b) + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot\boldsymbol{\nabla}\, b) + \frac{1}{Fr^2}\Big((w - \frac{1}{\Upsilon}w_s)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_{t\,z}\Big) = 0$$

Vertical velocity scales as $w \propto Fr^2 UD$, $D = H/L$, $Fr^2 = U^2/N^2H^2$, $\Phi = f_0 UL$, $Ro = U/(f_0 L), 1/Bu = Fr^2/Ro^2 = L^2/R_d^2$, $\Upsilon = (T \cdot MKE)/(T^\sigma \cdot TKE)$
Vertical momentum equation:

$$\frac{Ro}{Bu}D^2\Big(\mathrm{d}_t w + (\boldsymbol{u} - \frac{1}{\Upsilon}\boldsymbol{u_s}) \cdot\boldsymbol{\nabla}\, w\mathrm{d}t - \frac{1}{2\Upsilon}\, \boldsymbol{\nabla}\cdot (\boldsymbol{a}\boldsymbol{\nabla} w)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot\boldsymbol{\nabla}\, w\Big) = (-\partial_z\Phi + b)\mathrm{d}t - \partial_z \mathrm{d}\phi_t^\sigma$$

Hydrostatic balance must be discussed w.r.t $\frac{Ro}{Bu}D^2$ and the noise variance $1/\Upsilon$

Hydrostatic balance - I

Horizontal momentum:

$$Ro\big(\mathrm{d}_t \boldsymbol{u}^h + (\boldsymbol{u} - \frac{1}{\Upsilon}\boldsymbol{u_s}) \cdot\boldsymbol{\nabla}\ \boldsymbol{u}^h \mathrm{d}t - \frac{1}{2\Upsilon}\ \boldsymbol{\nabla}\cdot\ (\boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{u}^h)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot\boldsymbol{\nabla}\ \boldsymbol{u}^h\big) + \frac{1}{\Upsilon^{1/2}}\boldsymbol{f} \times (\boldsymbol{u}^h + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h) = -\boldsymbol{\nabla}^h \Phi \mathrm{d}t - \boldsymbol{\nabla}^h \mathrm{d}\Phi_t^\sigma$$

1 Noise regime $\frac{Ro}{Bu\Upsilon^{1/2}}D^2 << 1$ and $Ro/\Upsilon^{1/2} << 1$, $\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t)^h = 0$ (QG noise)

- Hydrostatic balance:

$$(-\partial_z\Phi + b)\mathrm{d}t - \partial_z \mathrm{d}\phi_t^\sigma = 0 \Longrightarrow (\underbrace{\partial_z\Phi = b}_{HB}) \text{ and } \partial_z \mathrm{d}\phi_t^\sigma = 0 \text{ i.e } \mathrm{d}\phi_t^\sigma(x,y)$$

- Martingale pressure term

$$\frac{1}{\Upsilon^{1/2}}\boldsymbol{f}\times\underbrace{\int_{-h}^{0}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h \mathrm{d}z}_{\widetilde{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}} = -\boldsymbol{\nabla}^h \mathrm{d}\Phi_t^\sigma h \Longrightarrow \mathrm{d}\Phi_t^\sigma = -\frac{f}{\Upsilon^{1/2}h}\Delta_h^{-1}(\boldsymbol{\nabla}\times\widetilde{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t})$$

Hydrostatic balance - II

Horizontal momentum:

$$Ro\Big(\mathrm{d}_t\boldsymbol{u}^h + (\boldsymbol{u} - \frac{1}{\Upsilon}\boldsymbol{u_s})\boldsymbol{\cdot\nabla}\ \boldsymbol{u}^h\mathrm{d}t - \frac{1}{2\Upsilon}\ \boldsymbol{\nabla\cdot}(\boldsymbol{a\nabla u}^h)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\boldsymbol{\cdot\nabla}\ \boldsymbol{u}^h\Big)+$$
$$\frac{1}{\Upsilon^{1/2}}\boldsymbol{f}\times(\boldsymbol{u}^h + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h) = -\boldsymbol{\nabla}^h\Phi\mathrm{d}t - \boldsymbol{\nabla}^h\mathrm{d}\Phi_t^\sigma$$

2 Noise regime $\frac{Ro}{Bu\Upsilon^{1/2}}D^2 << 1$ and $Ro/\Upsilon^{1/2} << 1$, $\boldsymbol{\nabla\cdot}(\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t)^h \neq 0$

- Hydrostatic balance:

$$(-\partial_z\Phi + b)\mathrm{d}t - \partial_z\mathrm{d}\phi_t^\sigma = 0 \Longrightarrow \underbrace{(\partial_z\Phi = b)}_{HB} \text{ and } \partial_z\mathrm{d}\phi_t^\sigma = 0 \text{ i.e } \mathrm{d}\phi_t^\sigma(x,y)$$

- Martingale pressure term

Helmholtz decomposition: $$\frac{1}{\Upsilon^{1/2}}\int_{-h}^{0}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h\mathrm{d}z = \frac{1}{\Upsilon_g^{1/2}}\widetilde{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}^g + \frac{1}{\Upsilon_{ag}^{1/2}}\widetilde{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}^{ag}$$

$$\frac{1}{\Upsilon^{1/2}}\boldsymbol{f}\times\widetilde{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t} = -\boldsymbol{\nabla}^h\mathrm{d}\Phi_t^\sigma h \Longrightarrow \mathrm{d}\Phi_t^\sigma = -\frac{f}{\Upsilon_g^{1/2}h}\Delta_h^{-1}(\boldsymbol{\nabla}\times\widetilde{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}^g)$$

Hydrostatic balance - III

Horizontal momentum:

$$Ro\Big(\mathrm{d}_t\boldsymbol{u}^h + (\boldsymbol{u} - \frac{1}{\Upsilon}\boldsymbol{u_s})\cdot\boldsymbol{\nabla}\,\boldsymbol{u}^h\mathrm{d}t - \frac{1}{2\Upsilon}\,\boldsymbol{\nabla}\cdot(\boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{u}^h)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\,\boldsymbol{u}^h\Big) + \frac{1}{\Upsilon^{1/2}}\boldsymbol{f}\times(\boldsymbol{u}^h + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h) = -\boldsymbol{\nabla}^h\Phi\mathrm{d}t - \boldsymbol{\nabla}^h\mathrm{d}\Phi_t^\sigma$$

3 Noise regime $\frac{Ro}{Bu\Upsilon^{1/2}}D^2 << 1$ and $\frac{Ro}{\Upsilon^{1/2}} \propto \mathcal{O}(1)$

- Hydrostatic balance:

$$(-\partial_z\Phi + b)\mathrm{d}t - \partial_z\mathrm{d}\phi_t^\sigma = 0 \Longrightarrow (\underbrace{\partial_z\Phi = b}_{HB}) \text{ and } \partial_z\mathrm{d}\phi_t^\sigma = 0 \text{ i.e } \mathrm{d}\phi_t^\sigma(x,y)$$

- Modified martingale geostrophic balance

$$\frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\,\boldsymbol{u}^h + \frac{1}{\Upsilon^{1/2}}\boldsymbol{f}\times\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h = -\boldsymbol{\nabla}^h\mathrm{d}\Phi_t^\sigma$$

- Martingale pressure term

$$\mathrm{d}\Phi_t^\sigma = -\frac{1}{h\Upsilon^{1/2}}\Delta^{-1}\,\boldsymbol{\nabla}\cdot\int_{-h}^{0}\big(\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\,\boldsymbol{u}^h + \boldsymbol{f}\times\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h\big)\mathrm{d}z$$

Hydrostatic balance - IV

Horizontal momentum:

$$Ro\Big(\mathrm{d}_t \boldsymbol{u}^h + (\boldsymbol{u} - \frac{1}{\Upsilon}\boldsymbol{u_s}) \boldsymbol{\cdot\nabla}\ \boldsymbol{u}^h \mathrm{d}t - \frac{1}{2\Upsilon}\ \boldsymbol{\nabla\cdot}\ (\boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{u}^h)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\ \boldsymbol{\cdot\nabla}\ \boldsymbol{u}^h\Big) + \frac{1}{\Upsilon^{1/2}}\boldsymbol{f} \times (\boldsymbol{u}^h + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t^h) = -\boldsymbol{\nabla}^h\Phi\mathrm{d}t - \boldsymbol{\nabla}^h\mathrm{d}\Phi_t^\sigma$$

4 Noise regime $\frac{Ro}{Bu\Upsilon^{1/2}}D^2 \propto \mathcal{O}(1)$

- Hydrostatic balance:

$$-\frac{1}{\Upsilon}\boldsymbol{u_s}\boldsymbol{\cdot\nabla} w\mathrm{d}t - \frac{1}{2\Upsilon}\boldsymbol{\nabla\cdot}(\boldsymbol{a}\boldsymbol{\nabla}w)\mathrm{d}t + \frac{1}{\Upsilon^{1/2}}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\boldsymbol{\cdot\nabla}w = (-\partial_z\Phi + b)\mathrm{d}t - \partial_z\mathrm{d}\phi_t^\sigma \Longrightarrow$$

- Pressure terms

$$\Phi(z) = \int_{-z}^{0}\Big(b + \frac{1}{2\Upsilon}\boldsymbol{u_s}\ \boldsymbol{\cdot\nabla}\ w + \frac{1}{2\Upsilon}\ \boldsymbol{\nabla\cdot}\ (\boldsymbol{a}\boldsymbol{\nabla}w)\Big)\mathrm{d}z'$$

$$\mathrm{d}\Phi_t^\sigma(z) = \frac{1}{\Upsilon^{1/2}}\int_{-z}^{0}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\ \boldsymbol{\cdot\nabla}\ w\mathrm{d}z'$$

$$w(z) = -\int_{-z}^{0}(\boldsymbol{\nabla\cdot}\boldsymbol{u}^h - \frac{1}{\Upsilon}\ \boldsymbol{\nabla\cdot}\ \boldsymbol{u_s})\mathrm{d}z'$$

Conclusions

- Stochastic modelling introduces a relaxation of the classical Hydrostatic assumption
- Schemes for the turbulent pressure computation depends on the noise regime
- High noise regime or high Rossby/Burger flow modifies the hydrostatic balance; it now depends on the Ito-Stokes drift and on the noise diffusion