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Long-distance propagation of tsunamis and surface waves: on the relative importance of dispersion and nonlinearity

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Outline

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Part II – **An assessment of dispersive effects for tsunami wave propagation**

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Part IV – **On solitary waves**

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Part I – Dispersion and nonlinearity for water waves: an introduction

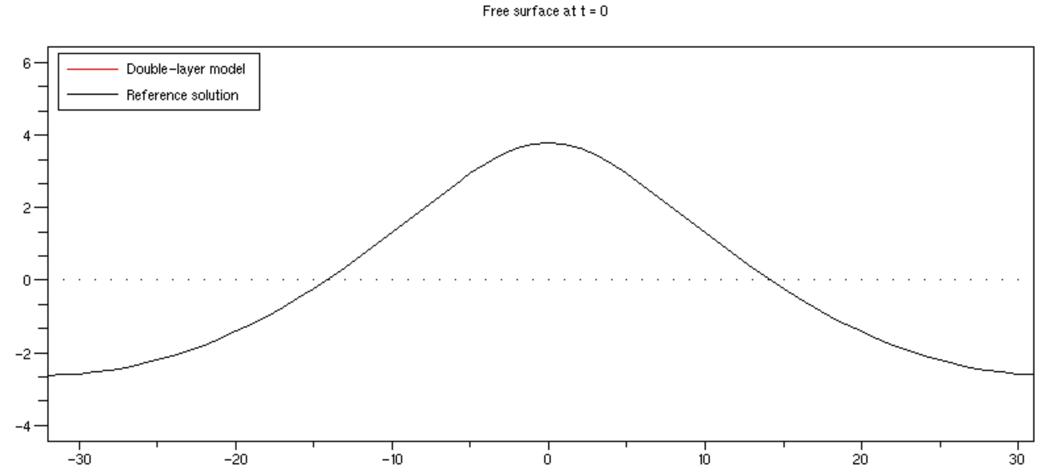
Nonlinear and dispersive waves on a flat bottom:

L = 64 m

H = 6.4 m **H/L = 10% (ka = π/10 = 0.31)**

depth h = 64 m **h/L = 1 (kh = 2π)**

No ambient current



What is dispersion and how does it manifest?

Dispersion refers to the form of the wave speed.

In the general case, the celerity C of a regular wave of wave-length L propagating in a domain of constant water depth h is a function of the wave period T and of the wave height H.

This is called the **dispersion relation**: $C = \frac{L}{T} = f(h, T, H)$

If this speed C is a function of the water depth h solely (irrespective of the period T and height H), waves are **non-dispersive**. They all propagate at the same speed, given by $C = \sqrt{gh}$. This is also called the "**long wave**" regime.

The shallow water equations (SWE) system is the prototype of non-dispersive models, for long waves with wave lengths much larger than the water depth. The criterion usually used to apply the long wave approx. is: L > 20 h or h/L < 1/20 or kh < π/10.

The effect of the period T (or frequency $f = 1/T$) on the celerity is referred to as **frequency dispersion**.

The effect of the wave height H (or amplitude $a = H/2$) on the celerity is referred to as **amplitude dispersion**.

How to quantify the (frequency) dispersion?

A way to consider (frequency) dispersion is related to the way the (surface) waves feel the proximity of the sea bottom.

Frequency dispersion is usually characterized by a non-dimensional ratio:

water depth / wave length

(is the wave long compared to the water depth?)

Parameters ususally used to evaluate (frequency) dispersion:

- the **relative water depth**: h/L or alternatively $\mu = kh = 2\pi h/L$ $(kh)^2$

- the **non-dimensional period**: $T\sqrt{\dfrac{g}{h}}$ or alternatively $\dfrac{gT^2}{h}$ $\dfrac{\omega^2 h}{g}$ $(\omega = 2\pi/T)$

Frequency dispersion in the linear wave theory

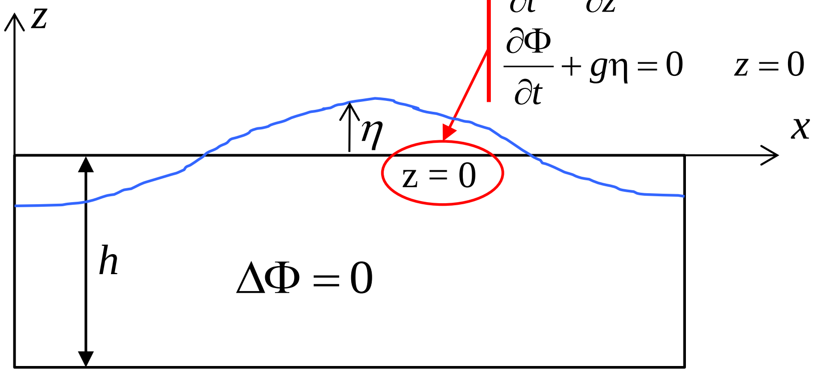
Consider the **Airy (or Stokes first order) linear wave model**, the prototype of linear (0 nonlinearity) and fully dispersive model for regular waves of period T.

$$\frac{\partial\eta}{\partial t}-\frac{\partial\Phi}{\partial z}=0 \qquad z=0 \quad \text{Kinematic FSBC}$$

$$\frac{\partial\Phi}{\partial t}+g\eta=0 \qquad z=0 \quad \text{Dynamic FSBC}$$

combined => $$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}=0 \qquad z=0$$

$$\eta=-\frac{1}{g}\frac{\partial\Phi}{\partial t}(z=0)$$



$$\frac{\partial\Phi}{\partial z}=0 \qquad z=-h$$

With periodic lateral BCs, this BVP Laplace problem on Φ has an analytical (exact) solution for progresive waves:

$$\Phi(x,z,t)=a\frac{g}{\omega}\frac{\cosh(k(z+h))}{\cosh(kh)}\sin\theta \qquad \theta=kx-\omega t \quad \text{Phase term}$$

$$\eta(x,t)=a\cos\theta$$

$$\boxed{\omega^2=gk\tanh(kh)}$$

Dispersion relation for linear water waves

The dispersion relation of linear wave theory

A fundamental relation for water waves: $\boxed{\omega^2 = gk\tanh(kh)}$

Alternative expressions:

- phase celerity: $C = \dfrac{L}{T} = \dfrac{\omega}{k} = \dfrac{g}{\omega}\tanh(kh) = \sqrt{\dfrac{g}{k}\tanh(kh)}$

- wave length: $L = \dfrac{gT^2}{2\pi}\tanh(kh)$

Remarks:

1. The speed C is function of water depth h and wave length L (or period T)
   => **frequency dispersion is present** !
2. The wave height H does not play any role: **no amplitude dispersion.**
   (which is fully coherent with the basic assumption of linear waves in this model).
3. Expressions with non-dimensional numbers:

$$\boxed{\chi = \mu\tanh\mu}$$

$$\chi = \frac{\omega^2 h}{g}$$

Non-dimensional frequency

$$\mu = kh$$

Relative water depth

$$\boxed{\frac{C}{\sqrt{gh}} = \sqrt{\frac{\tanh\mu}{\mu}}}$$

The dispersion relation – asymptotic cases

In $\omega^2 = gk\tanh(kh)$, $\mu = kh$ plays a dominant role through the tanh function.

If kh is large, $\tanh(kh) \approx 1$. The dispersion relation reduces to: $\omega^2 = gk_0$, $C_0 = g/\omega = gT/(2\pi)$

$\Rightarrow$ The celerity is a function of the wave period (or wave length) only. No effect of water depth h (which is coherent with the assumption that is depth is formally infinite).

$\Rightarrow$ **The waves are dispersive**: the longer waves travel faster.

$\Rightarrow$ "kh large" = L small (short waves) and/or h large (deep water).
This corresponds to the deep water or short wave regime.

$\Rightarrow$ Criterion for using this approximation: $h/L > 1/2 \leftrightarrow h > L/2 \leftrightarrow \mu = kh > \pi$

or in terms of nondimensional period: $T\sqrt{\dfrac{g}{h}} < 4$

If kh is small, $\tanh(kh) \approx kh$. The dispersion relation reduces to: $\omega^2 = ghk^2$, $C = \sqrt{gh}$

$\Rightarrow$ "kh small" = L large (long waves) and/or h small (shallow water).
This corresponds to the shallow water or long wave regime.

$\Rightarrow$ The celerity is a function of the water depth h only. No effect of wave period (or wave length): **the waves are non-dispersive**; they all propagate at the same speed.

$\Rightarrow$ Criterion for using this approximation: $h/L < 1/20 \leftrightarrow h < L/20 \leftrightarrow \mu = kh < \pi/10$

or in terms of nondimensional period: $T\sqrt{\dfrac{g}{h}} > 20$

Practical resolution of the dispersion relation (1)

$$\omega^2 = gk\tanh(kh)$$

This relation is **explicit** for computing ω or T as a function of h and L (or k).
(images of the sea surtfaces through radars
or photos at a given time => measure of L)

But it is **implicit** for computing L (or k) as a function of h and T (or ω).
(times series of free surface elevation at a given location
through buoy, pressure sensor, etc. => measure of T)

In this latter case, several options:

a. Use of asymptotic solutions if kh is large (> π) or small (< π/10), as seen previously,

b. Perform numerical (iterative) resolution,

c. Use of explicit approximations. Numerous expressions available in the literature.

d. Existing diagrams or tables with pre-calculated values of wave-length
as a function of water depth and period (in general less accurate).

Practical resolution of the dispersion relation (2)

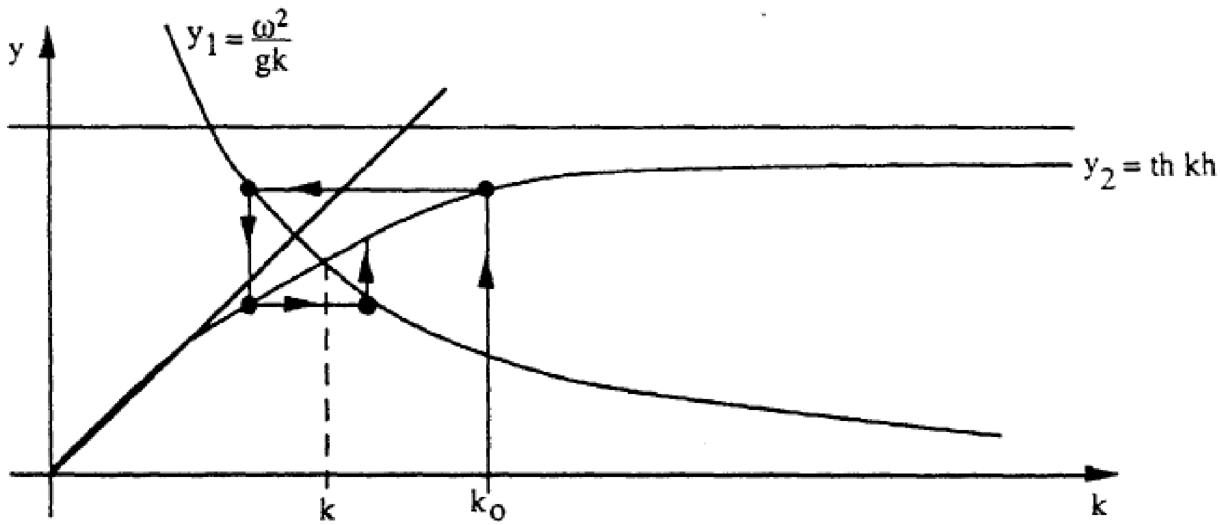
Numerical resolution with iterative scheme (fixed point iterations):

$$\omega^2 = gk\tanh(kh) \quad \Rightarrow \quad k_{(n)} = \frac{\omega^2}{g\tanh(k_{(n-1)}h)}$$

with initial guess: $k_{(0)} = \omega^2 / g$ if case closer to deep water,

$k_{(0)} = \omega / \sqrt{gh}$ if case closer to shallow water,

and a stopping criterion, e.g. $\left|k_{(n)} - k_{(n-1)}\right| < \varepsilon$



(Newton method can also be used, and other methods as well)

Practical resolution of the dispersion relation (3)

Explicit approximations:

Hunt (1979) (order 9)

$$(kh)^2 = (k_o h)^2 + \frac{k_o h}{1 + \sum_{n=1}^{9} a_n (k_o h)^n}$$

où $k_o = 2\pi/L_o = \omega^2/g$ = nombre d'onde au large (rad/m) et les valeurs de a_n sont les suivantes :

$a_1 = 0.66667$ $a_2 = 0.35550$ $a_3 = 0.16084$ $a_4 = 0.06320$ $a_5 = 0.02174$

$a_6 = 0.00654$ $a_7 = 0.00171$ $a_8 = 0.00039$ $a_9 = 0.00011.$

(very accurate – relative error < 0.1%)

Hunt (1979) (order 5)

$$(kh)^2 = (k_0 h)^2 + \frac{k_0 h}{1 + \sum_{n=1}^{5} a_n (k_0 h)^n}$$

$a_1 = 0{,}6522$ $a_2 = 0{,}4622$ $a_3 = 0$

$a_4 = 0{,}0864$ $a_5 = 0{,}0675$

Fenton & Mc Kee (1990)

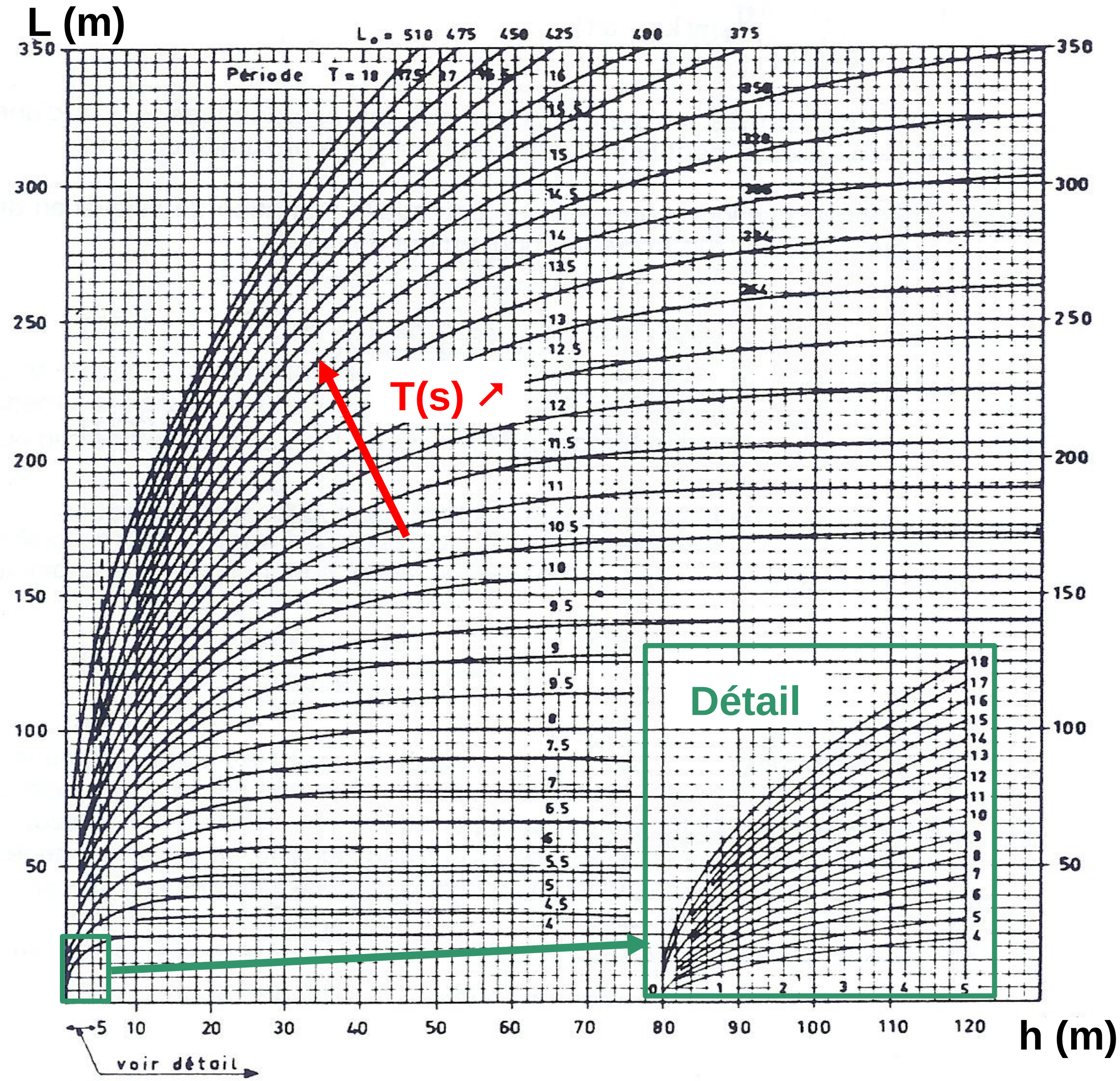
$$k = \frac{\omega^2}{g}\left\{\coth\left[\left(\omega\sqrt{\frac{h}{g}}\right)^{3/2}\right]\right\}^{2/3} \quad \text{ou} \quad L = L_o\left\{\tanh\left[(k_o h)^{3/4}\right]\right\}^{2/3}$$

(relative error < 1.5%)

Many other approximations (more or less complex and accurate).

Practical resolution of the dispersion relation (4)

Existing diagrams or tables with pre-calculated values



What is non-linearity and how does it manifest?

Nonlinearity refers to the effect of the wave amplitude (or height).

Nonlinear effects are present if the free surface deformation are finite (not infinitely small), and if the amplitude of the deformation plays a role in the evolution of the wave.

Conversely, in the linear framework, the amplitude of the wave is assumed to be infinitely small and do not play any role.

Nonlinearity is usually characterized by a non-dimensional ratio:

wave height / wave length or **wave height / water depth**
(is the wave high compared to its lenth or the water depth?)

Parameters ususally used to evaluate nonlinearity:

- the **steepness**: H/L or alternatively $\varepsilon = ka = \pi H/L$
  One often use the deep water wave length ($L_0 = gT^2/(2\pi)$) for the steepness: H/L_0
  This ratio is mostly used in deep water and intermediate water depth.

- the **relative wave height**: H/h or alternatively a/h.
  This ratio is mostly used in shallow water.

The larger these numbers, the stronger the nonlinear effects.

Another nondimensional number: the Ursell number

Ursell number: ratio nonlinearity / dispersion

$$Us = \frac{\text{steepness}}{(\text{relative depth})^3} = \frac{H/L}{(h/L)^3} = \frac{HL^2}{h^3} \qquad\qquad Us = \frac{H/h}{(h/L)^2}$$

with some variations, e.g. with *L = CT* and assuming, in shallow water $C = \sqrt{gh}$

$$Us^* = \frac{gHT^2}{h^2} = \frac{H/(gT^2)}{\left(h/(gT^2)\right)^2}$$

or with $a = H/2$:

$$Us^{**} = \frac{a/h}{(kh)^2} = \frac{H/L}{8\pi^2 (h/L)^3} = \frac{Us}{8\pi^2}$$

- Useful to quantify the level of non-linearities in the waves.

- Used for choosing an **analytical nonlinear wave theory** for regular waves on flat bottom.
Threshold value usually considered: **Us = 26**

Us < 26 => Stokes-type theories ; Us > 26 => cnoidal theories

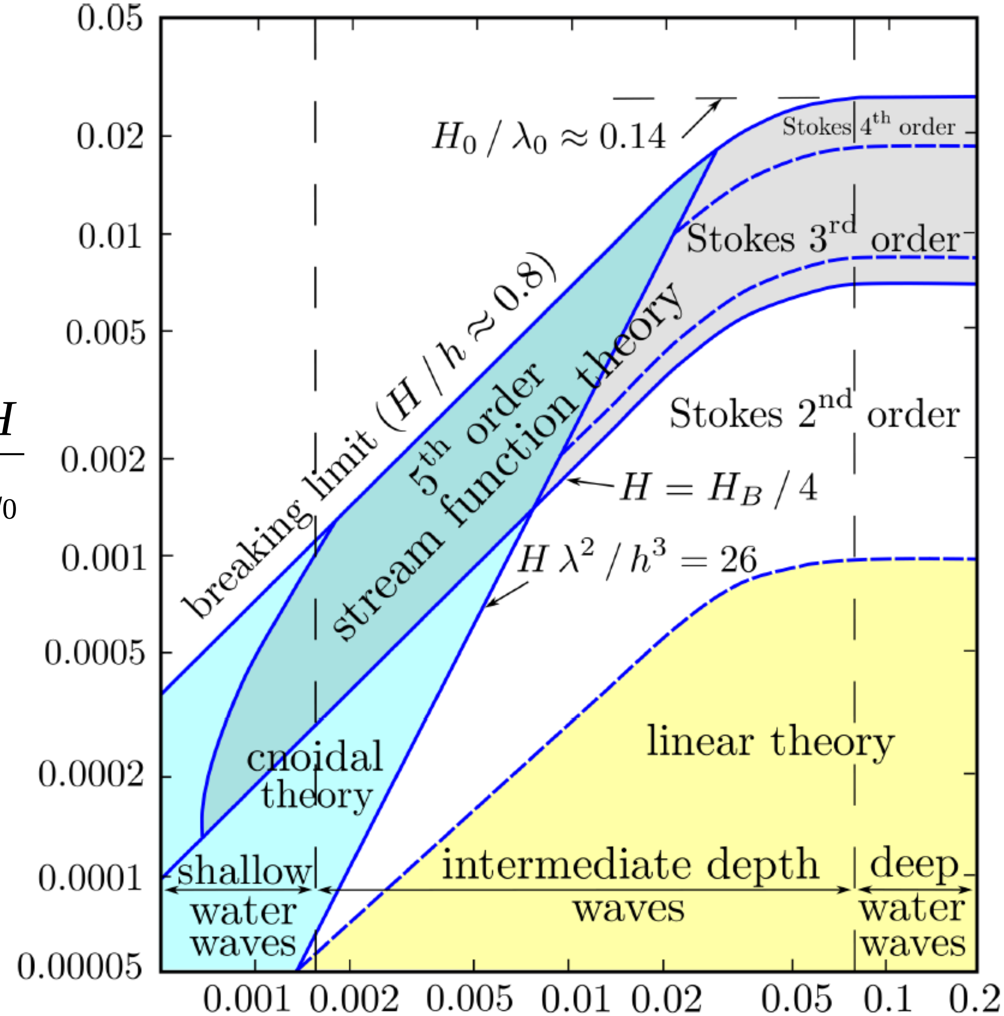
BUT it is advised to use a numerical method, like the **stream function Fourier series** method, valid whatever the relative water depth and the wave steepness.

Nonlinearity for regular (periodic) waves

When H/L or H/h exceed some thresholds the stability limit is reached and **breaking** occurs.

Example of periodic waves :
Le Mehaute diagram

Nonlinearity parameter (steepness) $\dfrac{H}{gT^2} = \dfrac{1}{2\pi}\dfrac{H}{L_0}$



Dispersion parameter (relative water depth) $\dfrac{h}{gT^2} = \dfrac{1}{2\pi}\dfrac{h}{L_0}$

Nonlinearity for regular (periodic) waves

Effect of wave height of the celerity (amplitude dispersion):

Example for a regular wave, with fixed characteristics:

Depth h = 100 m

Period T = 10 s

=> Effect of dispersion?

$T(h/g)^{0.5} = 3.13 < 4$ $h/(gT^2) \approx 0.1$ => deep water approx. is OK

=> wave length? $L_0 = gT^2/(2\pi) =$ 156.13 m ($L_0/2 = 78.06$ m)

(from linear dispersion relation; asymptotic deep water case)

We let the wave height H increase: H = 1 m, 8 m, 17 m,

and so do the steepness H/L and the Ursell number $U_s = \dfrac{HL^2}{h^3}$

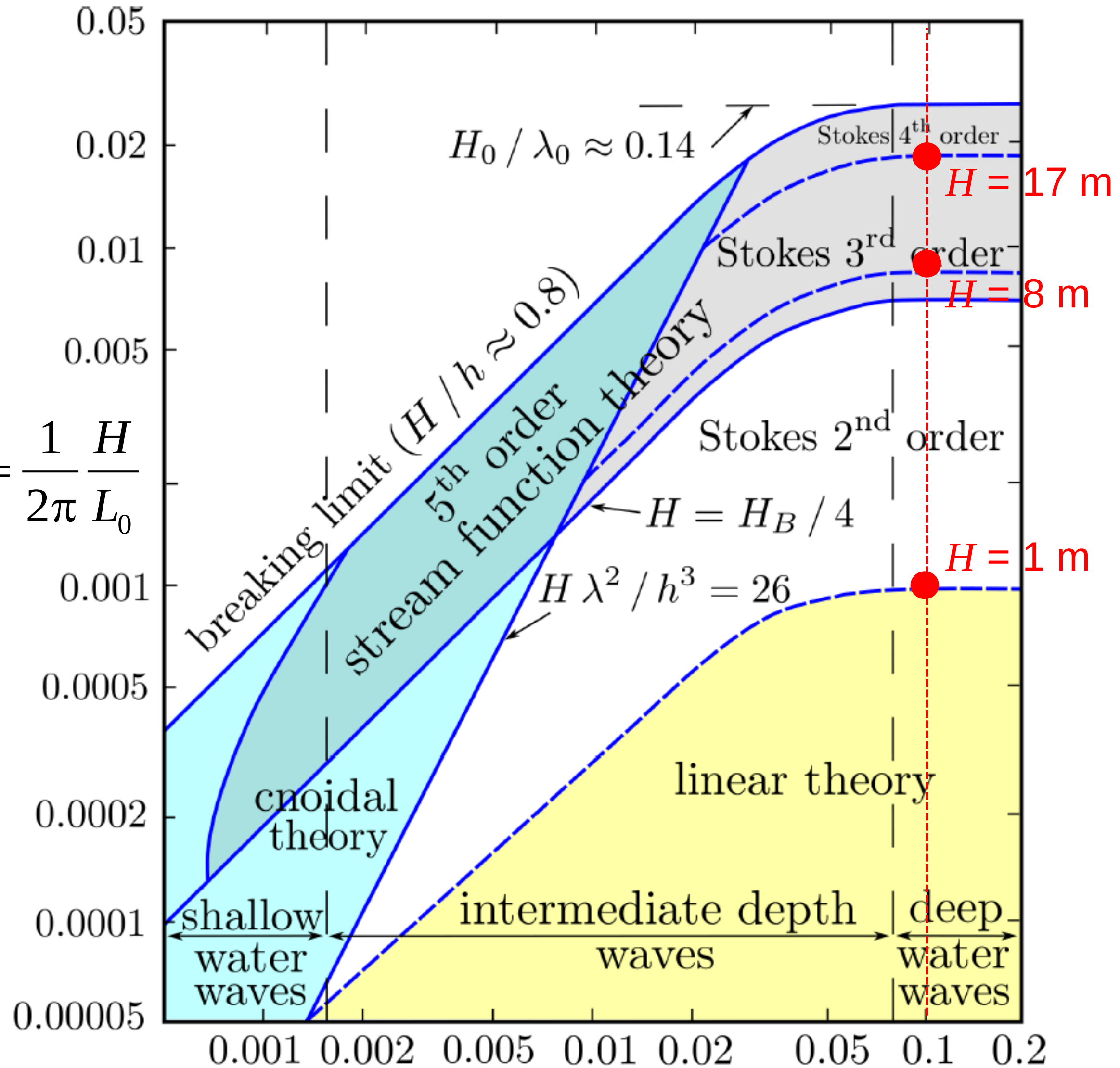
Compare the results from various wave theories:

Stokes 1 (=linear), Stokes 3, Stokes 5 and the stream function solution at order 18 (STREAM_HT code)

Nonlinearity for regular (periodic) waves

Nonlinearity parameter (steepness)

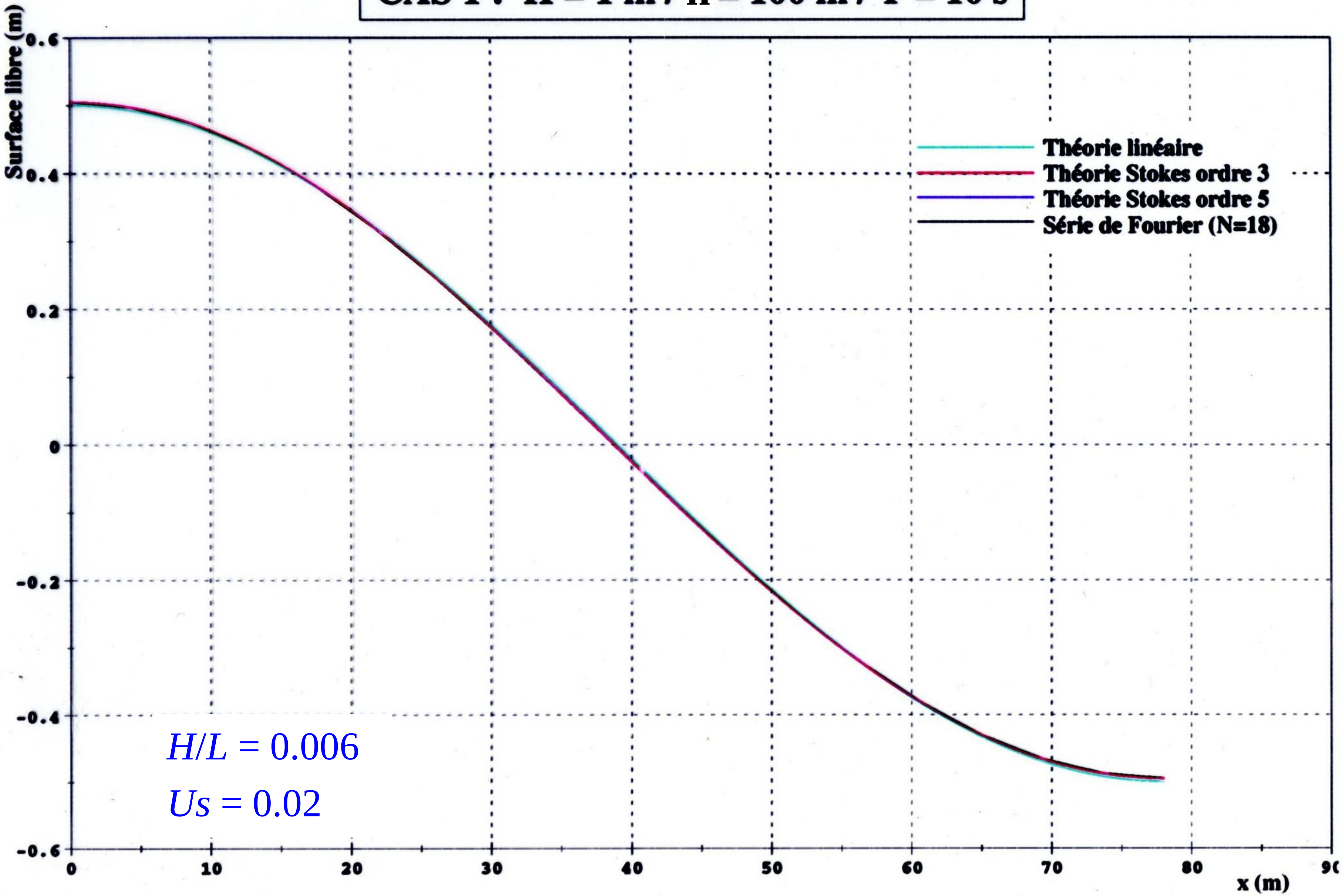
$$\frac{H}{gT^2} = \frac{1}{2\pi}\frac{H}{L_0}$$



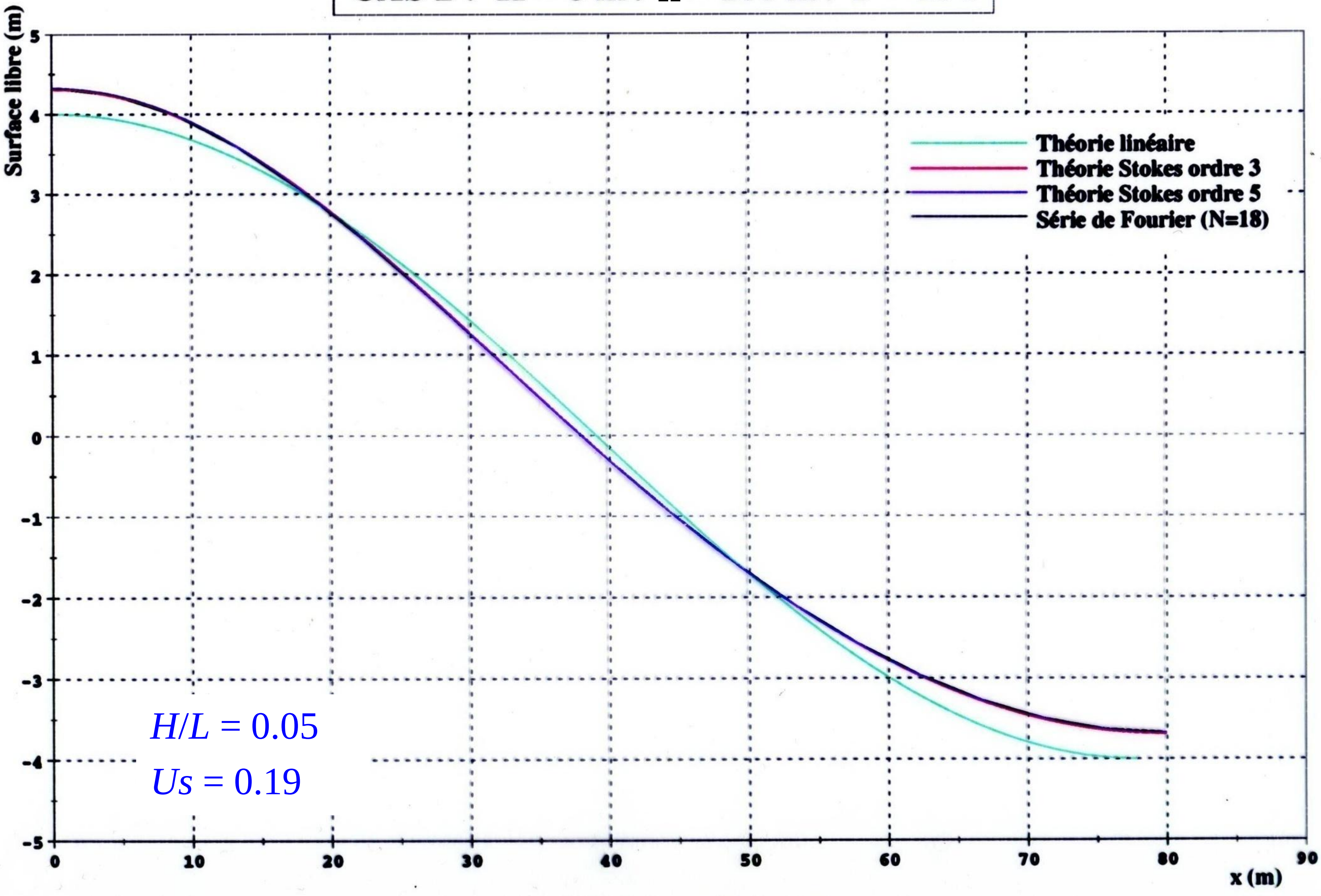
Dispersion parameter (relative water depth)

$$\frac{h}{gT^2} = \frac{1}{2\pi}\frac{h}{L_0}$$

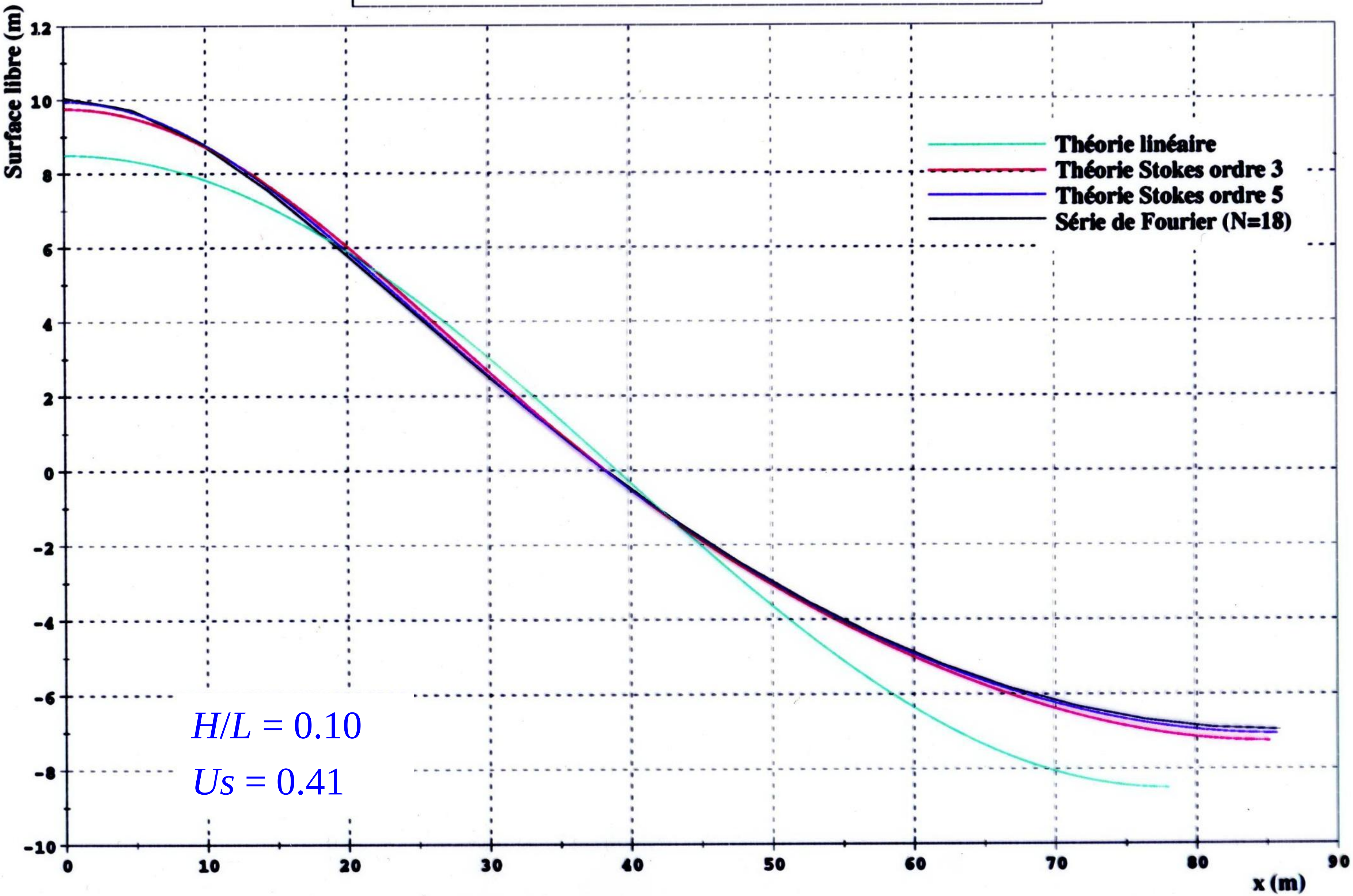
CAS 1 : H = 1 m / h = 100 m / T = 10 s



CAS 2 : H = 8 m / h = 100 m / T = 10 s



CAS 3 : H = 17 m / h= 100 m / T = 10 s



$H/L = 0.10$

$Us = 0.41$

Part II – An assessment of dispersive effects for tsunami wave propagation

A photo of the 2004 tsunami showing the « undular bore » shape of the tsunami approaching the coastline, with clear indication of the significance of dispersive and nonlinear effects.

A (very) simplified frame for the analysis

Assumptions:

- Flat bottom (h constant). Set $C_0 = (gh)^{1/2}$
- Linear weakly dispersive waves => linear KdV model $\eta_t + C_0 \eta_x + \cancel{\frac{3}{2}\frac{C_0}{h}\eta\eta_x} + \frac{1}{6} C_0 h^2 \eta_{xxx} = 0$
- Initial deformation of the free surface given and symmetric w.r.t the y axis, through F(x/λ)
- Consider waves propagating to the right only

See Whitham (1974), Mei et al. (2005), Madsen et al. (2008), Glimsdal et al. (2013)

Dispersion relation of the linear KdV eq: $C^2 = C_0^2 \left(1 - \frac{1}{3}(kh)^2 + O(kh)^4 \right)$

$$\omega = kC_0 \sqrt{1 - \frac{1}{3}(kh)^2} \quad \Longrightarrow \quad \omega \approx kC_0 \left(1 - \frac{1}{6}(kh)^2 \right)$$

Analytical solution can be formulated using the Airy function (Ai):

$$\eta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}_0(k) e^{\imath(kx - \omega(k)t)} \mathrm{d}k$$

with adim. variables $\xi = \frac{x - c_0 t}{\lambda}$, $\boxed{\tau = \frac{6 c_0 h^2 t}{\lambda^3}}$

$$\eta = G(\xi, \tau) = \frac{1}{2\pi} \int_0^{\infty} \hat{F}(s) e^{\imath\left(\xi s + \frac{1}{36}\tau s^3\right)} \mathrm{d}s \quad \Longrightarrow \quad \eta \sim \frac{\hat{F}(0)}{2(12\tau)^{\frac{1}{3}}} \mathrm{Ai}\left(\frac{\xi}{(12\tau)^{\frac{1}{3}}} \right)$$

Asymptotic solution for large times

Eléments sur la fonction d’Airy Ai(x) (source wikipedia)

La **fonction d'Airy Ai(z)** (et la fonction Bi(z) = fonction d'Airy de seconde espèce) sont des solutions de l'équation différentielle linéaire d'ordre 2 (équation d’Airy) :

$y'' - zy = 0$ dans le cas complexe ($y'' - xy = 0$ dans le cas réel)

Solution dans le cas où x est réel : $\mathrm{Ai}(x) = \dfrac{1}{\pi}\displaystyle\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\, dt$

(intégrale convergente)

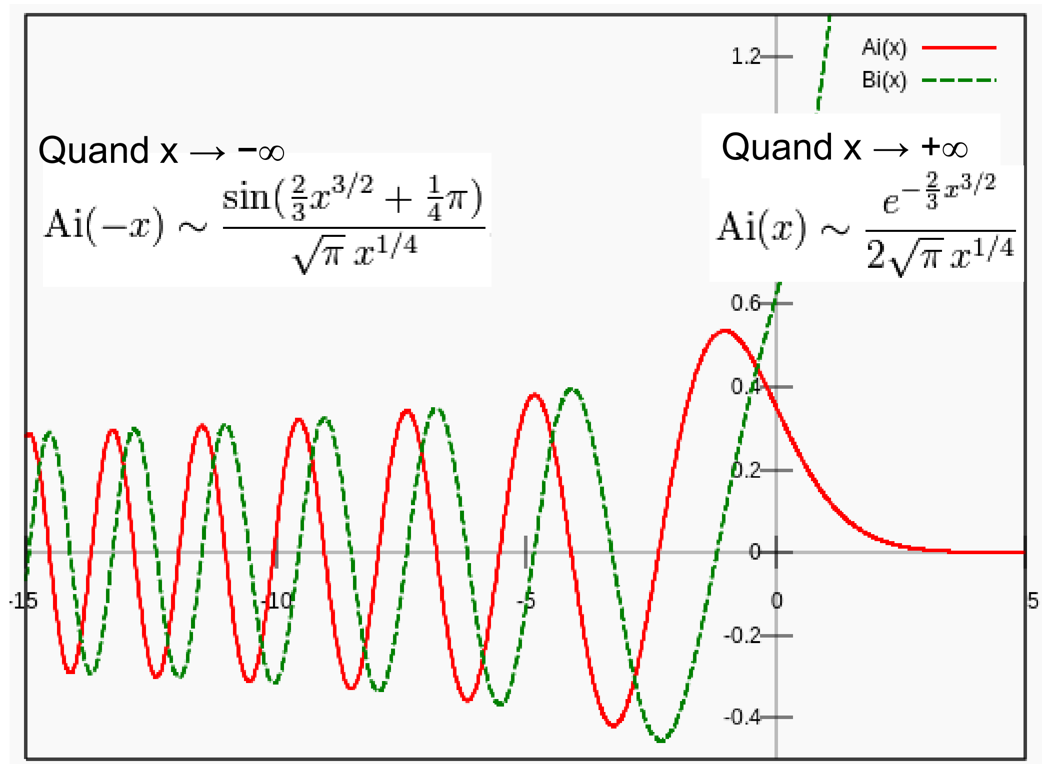
<u>Qq propriétés :</u>

•Pour x > 0, Ai(x) est positive, concave, et décroît exponentiellement vers 0.

•Pour x < 0, Ai(x) oscille autour 0 avec une fréquence de plus en plus forte et une amplitude de plus en plus faible à mesure que x → −∞)

•possède un point d'inflexion en x = 0

•Approximation quand x → +/−∞, voir sur graphe ci contre.



Interpretation of the dispersion time

Glimsdal et al. (2013) use the (nondimensional) **dispersion time** τ, as a measure of dispersive effects

$$\tau = \Delta c \cdot t \cdot \frac{1}{\lambda} \approx \frac{6c_0h^2}{\lambda^2} \cdot t \cdot \frac{1}{\lambda} = \frac{6h^2L}{\lambda^3} = \frac{6ht}{gT^3}$$

where

h is the water depth,

$c_0 = (gh)^{1/2}$ the long-wave celerity,

λ is a characteristic dimension of the initial wave length (of free surface initial deformation),

$T = \lambda/c_0$ the overall period,

$L = c_0 t$ the propagation distance in a time t,

$$\Delta c = c(0) - c(2\pi/\lambda) \approx 6c_0h^2/\lambda^2$$

The larger the dispersion time τ, the more significant are dispersive effects during the propagation.

- Dispersion accumulates in time, and thus increases with t and L.
- The variation is stronger with respect to h.
- The sensitivity is strongest with respect to the extension of the source λ

$$\boxed{\tau = \frac{6h^2L}{\lambda^3}}$$

Typical dispersion times

Glimsdal et al. (2013) have estimated dispersion times for several events, at a distance of L = 100 km from the source:

$$\tau = \frac{6h^2 L}{\lambda^3}$$

Location	M_w	B [km]	W [km]	D [deg]	$\bar{h}$ [km]	τ
SA, Andaman Islands	8.50	362	100	15	5	0.150 ←
SA, South Sumatra	9.10	527	200	15	5	0.019
BF, Myanmar - Bangladesh	8.90	655	125	10	3	0.028
MF, Pakistan coast	8.40	398	48	10	4	0.868 ←
BA, Eastern Banda Sea	8.50	261	150	20	3	0.016
NGT, Eastern Irian Jaya	8.50	258	100	20	4	0.096 ←
PT, South Mindanao	8.40	176	100	20	5	0.150 ←
MT, Western Luzon	8.20	348	70	45	5	0.437 ←
TT, Northern part	9.00	519	200	20	5	0.019
SST, Eastern Solomon Isl.	8.30	281	100	20	5	0.150 ←
NHT, Southern Vanuatu	8.60	314	100	20	3	0.054
NHT, Northern Vanuatu	8.60	340	100	20	3	0.054
PCT, Southern Chile	9.40	853	200	20	4	0.012 ←
PRT, North Hispaniola	8.00	200	55	80	5	0.902 ←
HA, South of Crete	7.70	149	75	20	3	0.128 ←

Table 2. Values for the dispersive parameter τ for a selection of sources applied in Løvholt et al. (2012a). τ is measured at a distance of 1000 km away from the source. The earthquake source parameters: M_w – magnitude; B – length; W – width; and D – dip angle. $\bar{h}$ is the average sea depth used for estimation of τ. Abbreviations: SA – Sunda Arc; BF – Burma Fault; MF – Makran Fault; BA – Banda Arc; NGT – New Guinea Trench; PT – Philippine Trench; TT – Tonga Trench; SST – South Solomon Trench; NHT – New Hebrides Trench; PCT – Peru–Chile Trench; PRT – Puerto Rico Trench; HA – Hellenic Arc.

Based on comparison with numerical simulations, they indicate that dispersive effects:

- are small (negligible) if $\tau < 0.01$, ←
- become significant for $\tau > 0.1$ ←

A summary of Glimsdal et al. (2013) study

$$\boxed{\tau = \frac{6h^2 L}{\lambda^3}}$$

The form of τ indicates that the source width (or initial wavelength for landslides) λ is more important for the significance of dispersion than the depth or propagation distance L.

=> moderate-magnitude earthquakes yield more dispersive tsunamis than the huge ones, such as the 2004 Indian Ocean and the 2011 Japan tsunami.

Seismic tsunamis (due to earthquakes):

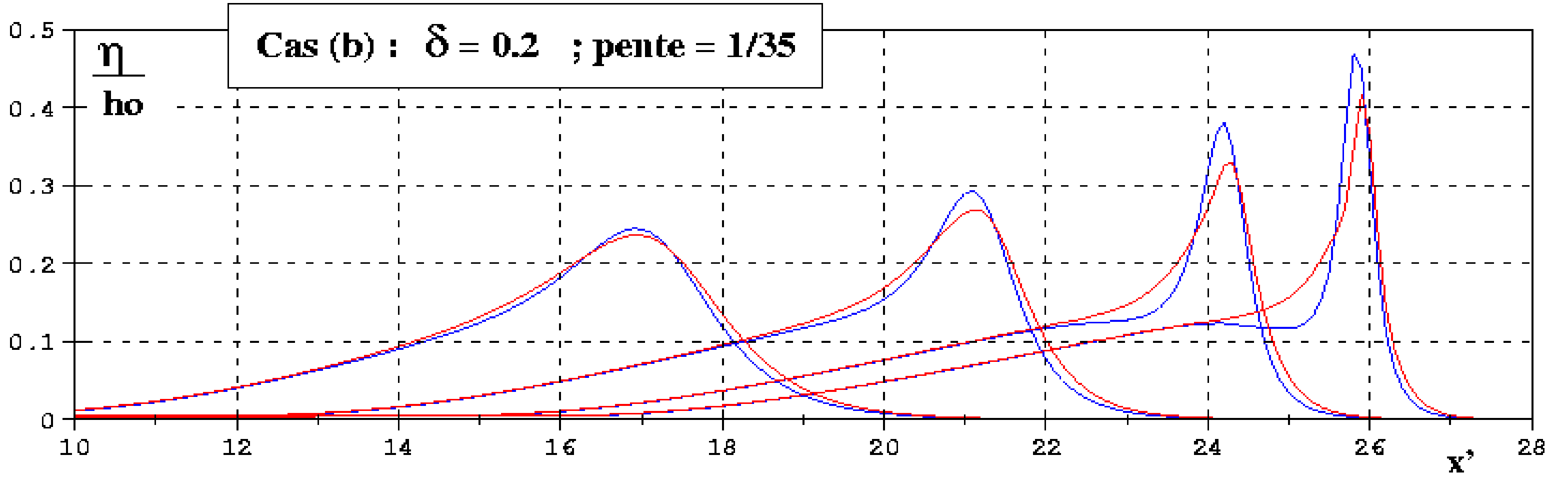
For the largest ones: frequency dispersion only modify the transoceanic propagation mildly. Hence, dispersion is not needed for propagation in the near-field, but may be important if far-field tsunami data are used for verification of source properties.

For the smaller ones (Mw < 8 or less): a strong directivity of the dispersion, following the amplitude directivity, due to the elongated shapes of the source regions. In the offshore direction normal to the fault line, the tsunami signal must be expected to become completely transformed before reaching buoys or other continents.

Landslide tsunamis:

Most of them are strongly affected by dispersive effects. For the leading part of the signal, such effects are generally most important during wave generation and the early stages of propagation. Extremely large landslides (moving at small Froude numbers), such as the Storegga Slide, are the likely exception. The oceanic propagation of such events is virtually non-dispersive.

Part III – Brief overview of numerical models for water waves and tsunamis



The shoaling of a solitary wave on a plane slope (1:35) simulated with two versions of Boussinesq models, up to the breaking point.

Non-dispersive nonlinear shallow water model

Saint-Venant equations (nonlinear shallow water equations) are expressed using the depth averaged horizontal velocity:

$$\underline{\overline{U}}(x,y,t) = \frac{1}{h+\eta}\int_{-h}^{\eta} \underline{u}(x,y,z,t)\,dz$$

rely on the following assumptions:

- hydrostatic distribution of pressure,
- constant velocity over the water column.

$$\frac{\partial \eta}{\partial t} + \nabla.\left((h+\eta)\underline{\overline{U}}\right) = 0 \qquad \textbf{Mass conservation}$$

$$\frac{\partial \underline{\overline{U}}}{\partial t} + \left(\underline{\overline{U}}.\nabla\right)\underline{\overline{U}} + g\nabla\eta = 0 \qquad \textbf{Momentum conservation}$$

Dispersive effects are neglected => suitable for long waves (tides, strorm surges)

Hyperbolic system (develop shocks from initial disturbances of the free surface).

On Boussinesq equations (a large family…)

Boussinesq equations from Peregrine (1967) for variable bottom profile:

$$\frac{\partial \eta}{\partial t} + \nabla.\left((h+\eta)\overline{\underline{U}}\right) = O(\delta^2, \delta\mu^2, \mu^4)$$

Orders of first neglected terms

$$\frac{\partial \overline{\underline{U}}}{\partial t} + \left(\overline{\underline{U}}.\nabla\right)\overline{\underline{U}} + g\nabla\eta + \boxed{\frac{h^2}{6}\nabla\left(\nabla.\frac{\partial \overline{\underline{U}}}{\partial t}\right) - \frac{h}{2}\nabla\left(\nabla.\left(h\frac{\partial \overline{\underline{U}}}{\partial t}\right)\right)} = O(\delta^2, \delta\mu^2, \mu^4)$$

Additional terms compared to Saint-Venant equations of order μ^2, representing dispersive effects

Expression for the 1DH case (along *x* only with velocity *U* in *x*) :

$$\eta_t + \left((h+\eta)U\right)_x = 0$$

$$U_t + g\eta_x + UU_x + \frac{h^2}{6}U_{xxt} - \frac{h}{2}\left(hU_t\right)_{xx} = 0$$

A number of higher order extensions have been introduced
(e.g. Nwogu, Madsen et al., Kirby et al., etc.)
=> a lot of variations of these equations, with more involved and complex expressions as higher order terms are added.

The Boussinesq equations of Nwogu (1993)

Unknowns: free surface η and horizontal speed u_α at elevation $z_\alpha = C_\alpha . h$

Nwogu (1993) equations in nondimensional variables $\mu = kh$ and $\delta = a/h$

- Mass conservation:

$$\frac{\partial \eta'}{\partial t'} + \nabla . \left((h' + \delta \eta') \vec{u}'_\alpha \right) + \mu^2 \nabla . \left[\left(\frac{z'^2_\alpha}{2} - \frac{h'^2}{6} \right) h' \nabla \left(\nabla . \vec{u}'_\alpha \right) + \left(z'_\alpha + \frac{h'}{2} \right) h' \nabla \left(\nabla . \left(h \vec{u}'_\alpha \right) \right) \right] = O(\delta \mu^2, \mu^4)$$

- Momentum conservation:

$$\frac{\partial \vec{u}'_\alpha}{\partial t'} + \nabla \eta' + \delta \left(\vec{u}'_\alpha . \nabla \right) \vec{u}'_\alpha + \mu^2 z'_\alpha \left[\frac{z'_\alpha}{2} \nabla \left(\nabla . \frac{\partial \vec{u}'_\alpha}{\partial t'} \right) + \nabla \left(\nabla . \left(h' \frac{\partial \vec{u}'_\alpha}{\partial t'} \right) \right) \right] = O\left(\delta \mu^2, \mu^4 \right)$$

Flat bottom case (in dimensional form):

$$\frac{\partial \eta}{\partial t} + h \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} (\eta u) + \left(\alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u}{\partial x^3} = 0 \qquad \text{with} \qquad \alpha = \frac{1}{2} \left(\frac{z_\alpha}{h} \right)^2 + \frac{z_\alpha}{h} = \frac{1}{2} C_\alpha^2 + C_\alpha$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} + u \frac{\partial u}{\partial x} + \alpha h^2 \frac{\partial^3 u}{\partial x^2 \partial t} = 0$$

Linear version of the model leads to the dispersion relation $\omega^2 = g k^2 h \dfrac{1 - \left(\alpha + \frac{1}{3} \right) (kh)^2}{1 - \alpha (kh)^2}$

NB: Equations of Wei et al. (1995) or Serre-Green-Naghdi (SGN) include all terms of order $O(\mu^2 \delta, \mu^2)$ => « fully nonlinear » equations (…at this μ^2 dispersive order however !)

Approximations of the dispersion relation of linear waves on a flat bottom

Taylor expansions at low values of μ = kh:

$$\text{Taylor-2}: \quad \left(\frac{C}{\sqrt{gh}}\right)^2 = \frac{\tanh(kh)}{kh} \approx 1 - \frac{1}{3}(kh)^2 + O(\mu^4)$$

KdV model

$$\text{Taylor-4}: \quad \left(\frac{C}{\sqrt{gh}}\right)^2 = \frac{\tanh(kh)}{kh} \approx 1 - \frac{1}{3}(kh)^2 + \frac{2}{15}(kh)^4 + O(\mu^6)$$

Padé(0,2N) rational aproximations

$$\text{Padé}(0,2): \quad \left(\frac{C}{\sqrt{gh}}\right)^2 = \frac{\tanh(kh)}{kh} \approx \frac{1}{1 + \frac{1}{3}(kh)^2} + O(\mu^4)$$

$$\text{Padé}(0,4): \quad \left(\frac{C}{\sqrt{gh}}\right)^2 = \frac{\tanh(kh)}{kh} \approx \frac{1}{1 + \frac{1}{3}(kh)^2 - \frac{1}{45}(kh)^4} + O(\mu^6)$$

Padé(2N,2N) rational aproximations:

$$\text{Padé}(2,2): \quad \left(\frac{C}{\sqrt{gh}}\right)^2 = \frac{\tanh(kh)}{kh} \approx \frac{1 + \frac{1}{15}(kh)^2}{1 + \frac{2}{5}(kh)^2} + O(\mu^6)$$

Madsen *et al* (1991)

$$\text{Padé}(4,4): \quad \left(\frac{C}{\sqrt{gh}}\right)^2 = \frac{\tanh(kh)}{kh} \approx \frac{1 + \frac{1}{9}(kh)^2 + \frac{1}{945}(kh)^4}{1 + \frac{4}{9}(kh)^2 + \frac{1}{63}(kh)^4} + O(\mu^{10})$$

Madsen *et al* (1998)

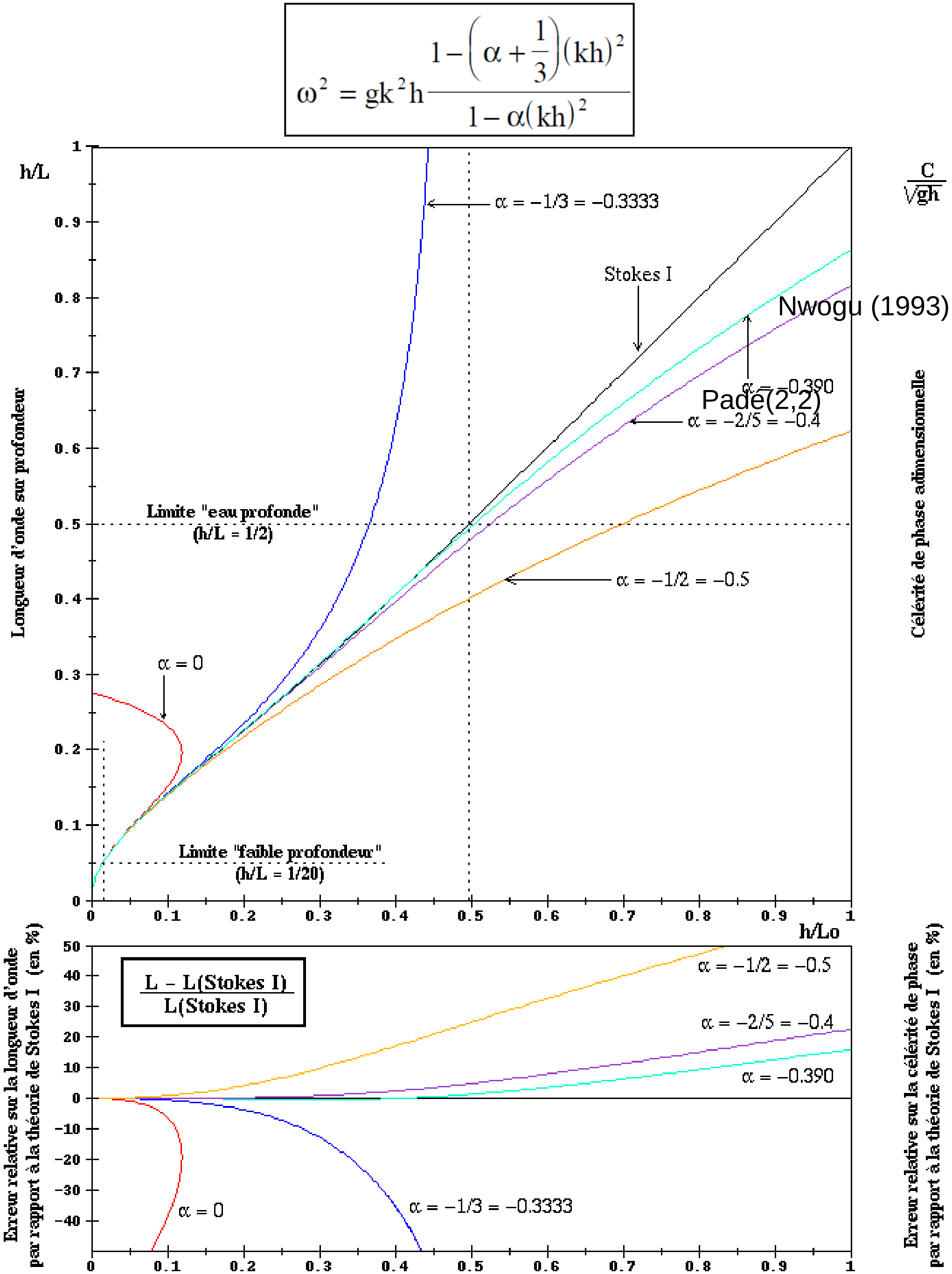


Figure 1 : Longueur d'onde des équations de Boussinesq linéarisées avec comparaison à la théorie de Stokes I (Airy)

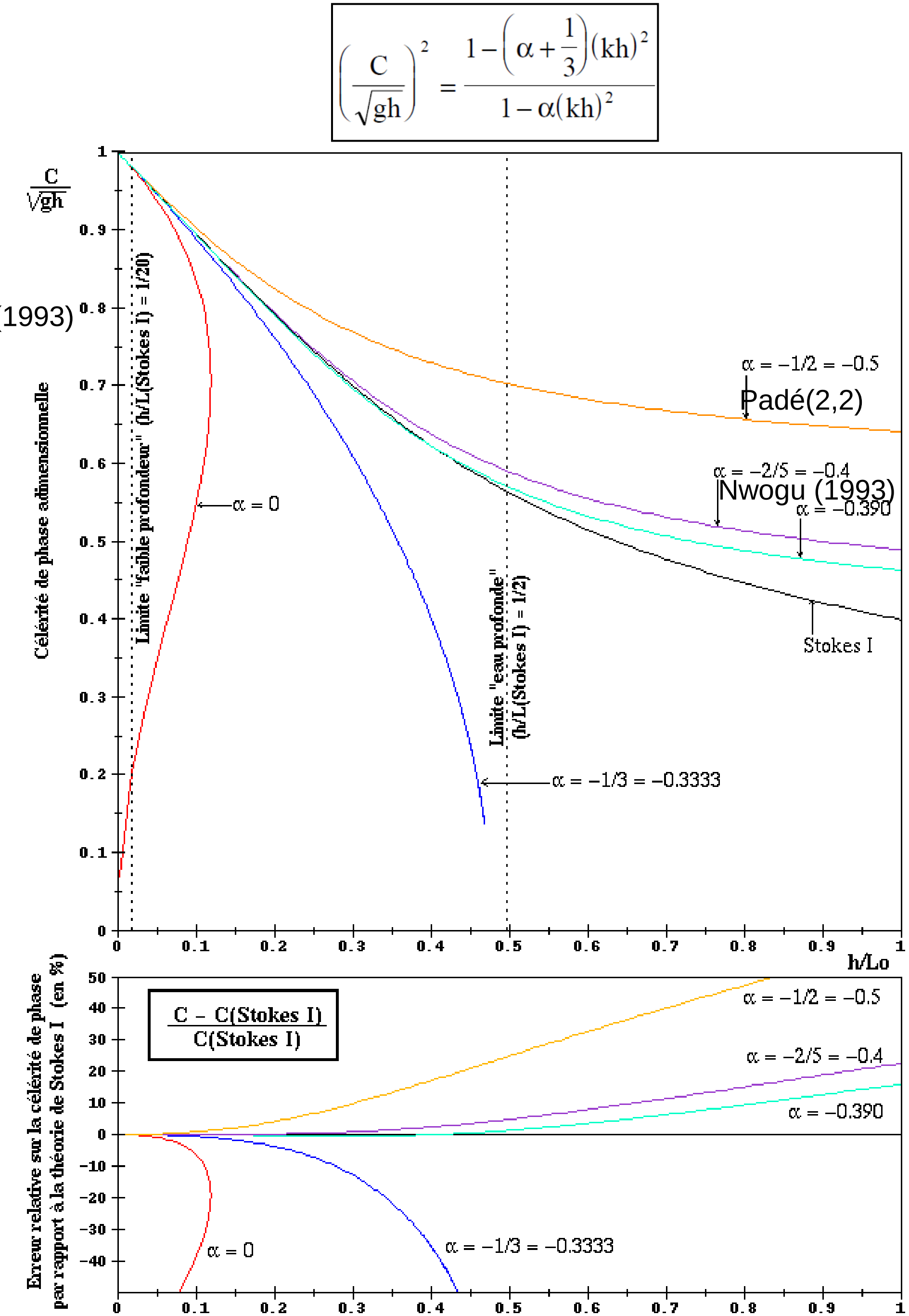


Figure 2 : Célérité de phase des équations de Boussinesq linéarisées avec comparaison à la théorie de Stokes I (Airy)

A fully nonlinear and dispersive model - Assumptions

Main assumptions of the mathematical model:

{H1} homogeneous fluid of constant density ρ (incompressible flow)

{H2} irrotational flow => wave potential $\phi(\underline{x},z,t)$: $\underline{u}(\underline{x},z,t) = \nabla\phi$ with $\underline{x} = (x,y)$

{H3} inviscid fluid, and no dissipative processes are included (e.g. depth-induced breaking, bottom friction), **but** some viscous effects can be included (see later).

{H4} non-overturning waves => continuous water column between bottom and free surface, free surface elevation η is a single-valued function of $\underline{x}$.

Remarks:

- **surface tension effects can be included** (=> gravity-capillary waves)
- **atmospheric pressure at the free surface can vary in $\underline{x}$ and t,** but will be taken here as homogeneous and constant (P_{atm} = 0 by convention)
- **bottom elevation $z = -h(\underline{x},t)$ can vary in $\underline{x}$ and t** (=> tsunami waves)

No assumptions made in the derivation of the model for:

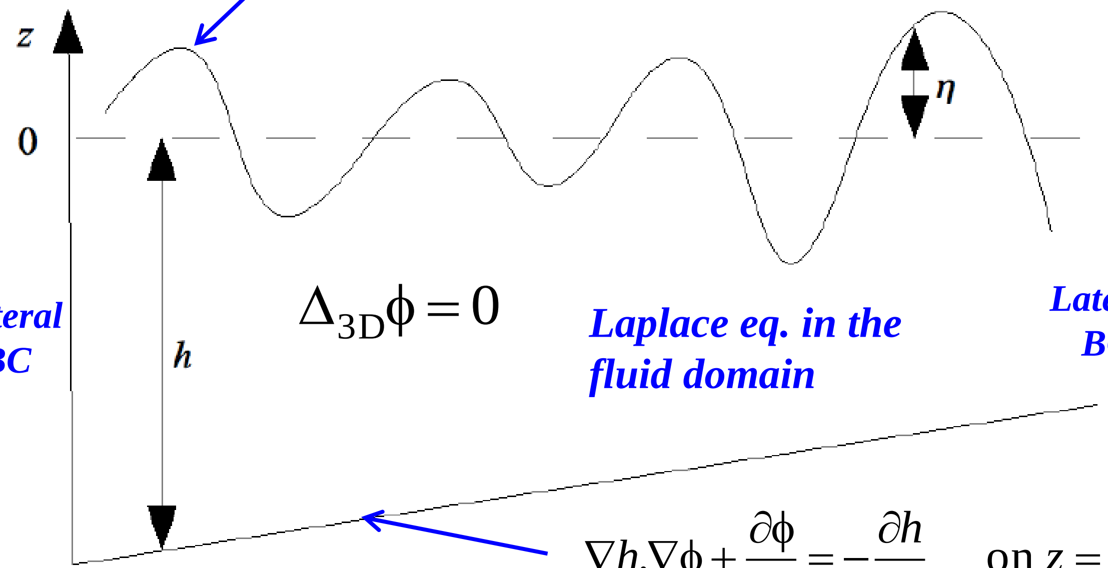
- **dispersive effects:** arbitrarily high values of $\mu = kh$ (deep to shallow water)
- **nonlinear effects:** arbitrarily high values of $\varepsilon = H/h$ (or H/L) (up to limit of stability)
- **bottom slope effects:** arbitrarily high values of $m = |\nabla h|$ (up to near-vertical slopes)

Mathematical model – Basic equations

Boundary conditions at the free surface $z = \eta(\underline{x},t)$

$$\frac{\partial \eta}{\partial t} + \nabla\eta.\nabla\phi - \frac{\partial \phi}{\partial z} = 0 \qquad \textbf{\textit{KFSBC}}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 + g\eta - \frac{\sigma}{\rho}\nabla.\left(\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}\right) = 0 \qquad \textbf{\textit{DFSBC}}$$



Lateral BC

$$\Delta_{3D}\phi = 0$$

Laplace eq. in the fluid domain

Lateral BC

$$\nabla h.\nabla\phi + \frac{\partial \phi}{\partial z} = -\frac{\partial h}{\partial t} \qquad \text{on } z = -h(\underline{x},t)$$

Boundary condition at the bottom

Non-hydrostatic pressure from Bernoulli eq.: $p(\underline{x},z,t) = -\rho\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 + gz\right)$

Mathematical model – Zakharov equations and DtN problem (1)

Introduce the free surface potential $\tilde{\varphi}(\underline{x},t) \equiv \phi(\underline{x}, z=\eta(\underline{x},t),t)$

(NB: a function of $\underline{x}$ and t only)

The 2 free-surface BC are written as the so-called **Zakharov equations** for $\eta(\underline{x},t)$ and $\tilde{\varphi}(\underline{x},t)$

$$\boxed{\begin{cases} \dfrac{\partial \eta}{\partial t} = -\nabla\tilde{\varphi}.\nabla\eta + \tilde{w}\left(1+(\nabla\eta)^2\right) \\ \dfrac{\partial \tilde{\varphi}}{\partial t} = -g\eta - \dfrac{1}{2}(\nabla\tilde{\varphi})^2 + \dfrac{1}{2}\tilde{w}^2\left(1+(\nabla\eta)^2\right) + \dfrac{\sigma}{\rho}\nabla.\left(\dfrac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}\right) \end{cases}}$$

$\tilde{w}(\underline{x},t) = \dfrac{\partial\phi}{\partial z}(z=\eta)$ is the vertical velocity at the free surface.

NB: all variables in this system are functions of $\underline{x}$ and t only in the general 3D case.

Zakharov (1968) has shown that **this system is Hamiltonian**,

with Hamiltonian = total energy $\boldsymbol{H = K + U}$

$$K = \frac{1}{2}\int d\underline{x}\int_{-h}^{\eta}\left((\nabla\phi)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right)dz \qquad U = \frac{g}{2}\int \eta^2 d\underline{x} + \frac{\sigma}{\rho}\int\left[\sqrt{1+(\nabla\eta)^2} - 1\right]d\underline{x}$$

Mathematical model – Zakharov equations and DtN problem (2)

To integrate in time the Zakharov system
$$\begin{cases} \dfrac{\partial \eta}{\partial t} = -\nabla\tilde{\varphi}.\nabla\eta + \tilde{w}\left(1+(\nabla\eta)^2\right) \\ \dfrac{\partial \tilde{\varphi}}{\partial t} = -g\eta - \dfrac{1}{2}(\nabla\tilde{\varphi})^2 + \dfrac{1}{2}\tilde{w}^2\left(1+(\nabla\eta)^2\right) + \dfrac{\sigma}{\rho}\nabla.\left(\dfrac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}\right) \end{cases}$$

at each (sub) time step, a **Dirichlet-to-Neumann** (DtN) problem needs to be solved:

given $\eta(\underline{x},t)$ and $\tilde{\varphi}(\underline{x},t)$, evaluate $\tilde{w}(\underline{x},t) = \left.\dfrac{\partial \phi}{\partial z}\right|_{z=\eta(\underline{x},t)}$ (**DtN** problem)

Existing approaches:

High Order Spectral (HOS) method (West *et al*., 1987 ; Dommermuth & Yue, 1987)

- rectangular domains with constant depth and periodic (or fully reflective) lateral BCs,
- extensions to irregular bottom (Smith, 1998 ; Craig *et al*., 2005, Guyenne & Nicholls, 2007).

Boussinesq-type models :

- approximate models from the Zakharov eq. (e.g. Madsen *et al.*, 2002, 2003, 2006)
- double-layer model of Chazel *et al*. (2009, 2010)

See also (among others) Kennedy & Fenton (2001) [Local Polynomial Approx.]
Clamond & Grue (2001), Fructus *et al*. (2005),
Bingham & Zhang (2007), Engsig-Karup *et al.* (2009).

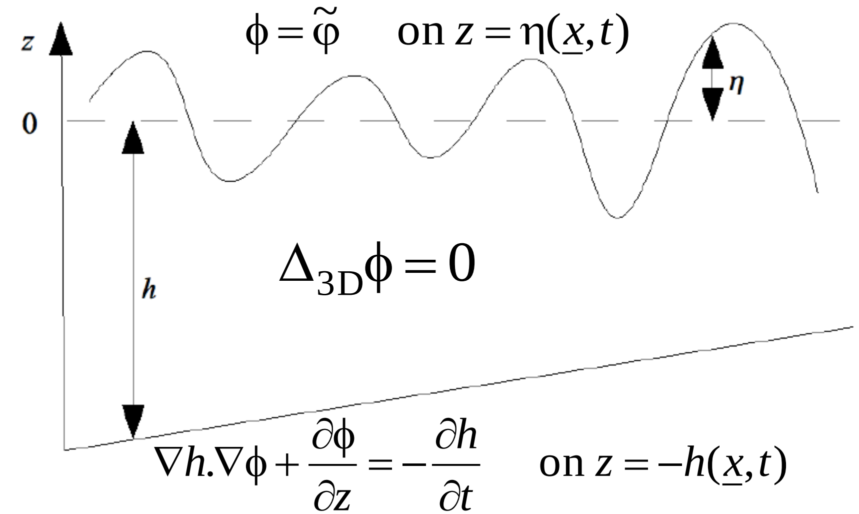
Overview of the selected numerical strategy

$$\begin{cases} \dfrac{\partial \eta}{\partial t} = -\nabla\tilde{\varphi}.\nabla\eta + \tilde{w}\left(1+(\nabla\eta)^2\right) \\ \dfrac{\partial \tilde{\varphi}}{\partial t} = -g\eta - \dfrac{1}{2}(\nabla\tilde{\varphi})^2 + \dfrac{1}{2}\tilde{w}^2\left(1+(\nabla\eta)^2\right) + \dfrac{\sigma}{\rho}\nabla.\left(\dfrac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}\right) \end{cases}$$

=> govern temporal evolution of free surface variables η and $\tilde{\varphi}$

Time marching: explicit 4th order Runge-Kutta (RK4) scheme, with constant Δt.
Alternative schemes will be tested, in particular symplectic schemes (Xu & Guyenne, 2009)

DtN problem: at each time step, solve the Laplace BVP numerically to compute $\phi(\underline{x},z,t)$ in the whole domain, and then compute $\tilde{w}(\underline{x},t)$ at the free surface.



➢ method for solving the Laplace BVP:
- in the vertical: spectral approach,
- in the horizontal:
  1DH: high-order finite difference scheme in x direction (Fornberg, 1988).
  2DH: discretization with RBF (Radial Basis Functions), currently ongoing.

➢ Misthyc code (collaboration IRPHE and Saint-Venant Lab.)

Solve Laplace BVP with spectral approach in the vertical (1)

Following Tian & Sato (2008) and Yates & Benoit (2015):

1/ Map vertical coordinate z on $[-h(x,t)\,;\,\eta(x,t)]$ to s on [-1 ; 1] ➔ rectangular domain

$$s(x,z,t) = \frac{2z + h^-(x,t)}{h^+(x,t)} \qquad s \in [-1\,;\,1] \qquad \begin{aligned} h^+(x,t) &\equiv h(x,t) + \eta(x,t) \\ h^-(x,t) &\equiv h(x,t) - \eta(x,t) \end{aligned}$$

2/ Reformulate the BVP problem in this transformed space (x, s) for $\phi(x,z,t) \equiv \varphi(x,s(x,z,t),t)$

$$\begin{aligned} \varphi_{xx} + 2s_x\varphi_{xs} + (s_x^2 + s_z^2)\varphi_{ss} + s_{xx}\varphi_s &= 0 && \text{in the fluid domain,} \\ h^+h_x\varphi_x + 2(1+h_x^2)\varphi_s &= 0 && \text{for } s = -1, \\ \varphi(x,1) &= \tilde{\Phi}(x) && \text{for } s = 1. \end{aligned}$$

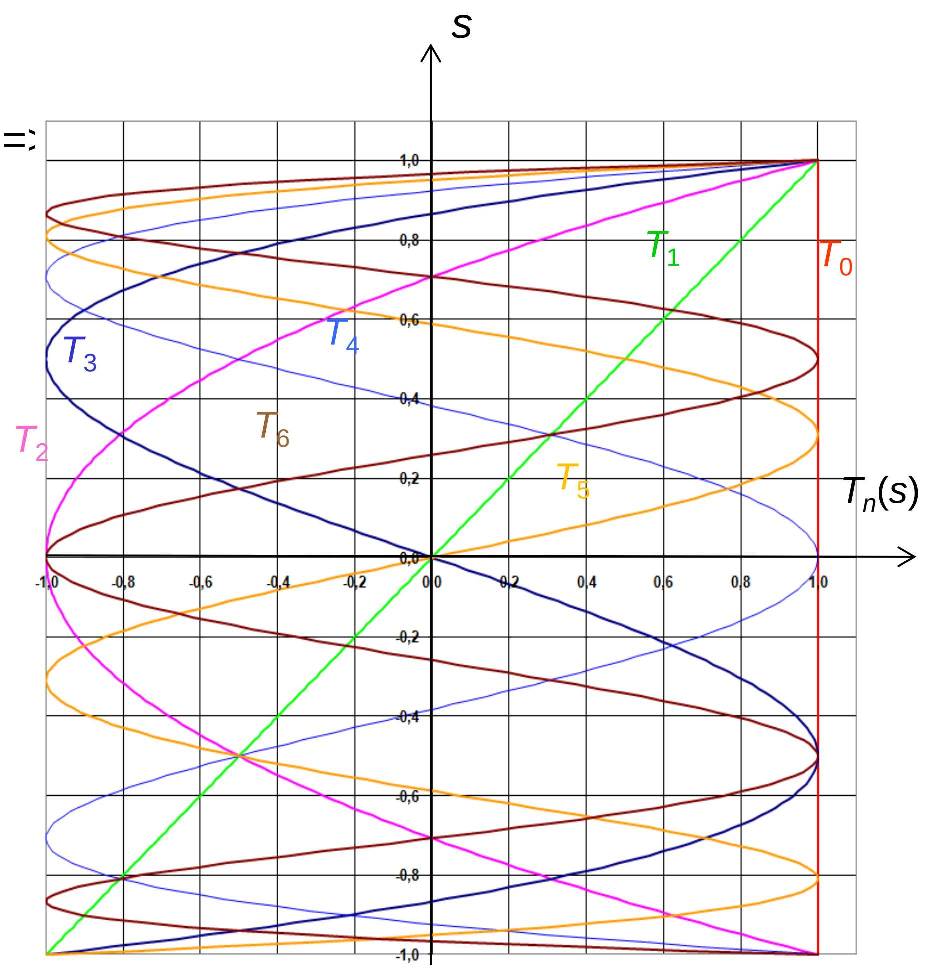
3/ Spectral approach in the vertical: series expansion of the potential in the vertical using Chebyshev polynomials of the first kind $T_n(s)$ (n = 0,..., N_T):

$$\varphi(x, s(x,z)) \approx \sum_{n=0}^{N_T} a_n(x) T_n(s) \qquad \text{Maximal order } N_T$$

=> $N_T + 1$ unknowns at each position: $a_n(x)$ (n = 0,..., N_T)

Solve Laplace BVP with spectral approach in the vertical (2)

Chebyshev polynomials of the first kind $T_n(s)$ =>



With inner product:

$$< f, g > \equiv \int_{-1}^{1} \frac{f(s)g(s)}{\sqrt{1-s^2}} ds$$

the $T_n(s)$ form an orthogonal basis on [-1 ; 1]:

$$< T_n, T_p > = \begin{cases} 0 & \text{if} \quad n \neq p, \\ \pi & \text{if} \quad n = p = 0, \\ \frac{\pi}{2} & \text{if} \quad n = p \neq 0. \end{cases}$$

Define the operator:

$$< f >_p \equiv \frac{2}{\pi C_p} < f, T_p >, \quad \text{with} \begin{cases} C_0 = 2, \\ C_p = 1 \quad \text{for} \quad p > 0. \end{cases}$$

$$\boxed{< \ T_n \ >_p = \ \delta_{np}}$$

Solve Laplace BVP with spectral approach in the vertical (3)

4/ Insert this approximation in Laplace equation:

(*) $$\sum_{n=0}^{N_T}[a''_n T_n + 2s_x a'_n T'_n + a_n((s_x^2 + s_z^2)T''_n + s_{xx}T'_n)] = 0$$

with
$$s_x = (h_x^- - s h_x^+)/h^+, \quad s_z = 2/h^+, \quad s_{zz} = 0,$$
$$s_{xx} = [(h_{xx}^- h^+ - 2h_x^- h_x^+) + s(2(h_x^+)^2 - h_{xx}^+ h^+)]/(h^+)^2,$$
$$a'_n \equiv \frac{da_n}{dx}, \quad a''_n \equiv \frac{d^2a_n}{dx^2}, \quad T'_n \equiv \frac{dT_n}{ds} \quad \text{and} \quad T''_n \equiv \frac{d^2T_n}{ds^2}.$$

5/ Use Chebyshev-tau method to remove *s*-dependence.

The operator $<\ldots>_p$ is applied to (*) for p = 0,….., N_T - 2 => $N_T - 1$ equations

$$a''_p + \sum_{n=0}^{N_T} C_{pn} a'_n + \sum_{n=0}^{N_T} D_{pn} a_n = 0, \quad p = 0, 1, ..., N_T - 2,$$

Terms C_{pn} and D_{pn} depend on $h(x,t)$, $\eta(x,t)$ and their 1st and 2nd derivatives in space, and the constant coefficients

$$B_{pikn} \equiv < s^i \frac{\partial^k T_n}{\partial s^k}(s) >_p$$

which can be computed analytically (once, at the start of the run).

Solve Laplace BVP with spectral approach in the vertical (4)

6/ Boundary conditions provide 2 equations at each position:

Bottom BC at s = -1
$$\sum_{n=0}^{N_T}[(-1)^n h^+ h_x a'_n + (-1)^{n-1} 2n^2(1+h_x^2)a_n] = 0$$

Dirichlet free-surface BC at s = +1
$$\sum_{n=0}^{N_T} a_n = \tilde{\Phi}.$$

<u>Summary:</u>
At each node x, there are N_T + 1 unknowns (a_0, a_1,….., a_{NT}) and we have formed N_T + 1 equations
- Galerkin form of Laplace eq. for p = 0, 1, …, N_T – 2
- impermeability condition at the bottom
- Dirichlet condition on the potential at the free surface.

For each node x_i (i = 1,…,NPX), system of N_T+1 linear equations on coefficients a_n (n = 0,..,N_T)
➔ sparse quare matrix of size **NPX*(N_T+1)**
➔ direct linear solver MUMPS (Amestoy et al., 2001, 2006).

Then the vertical velocity at free surface is
$$\tilde{w}(x) = \phi_s s_z\big|_{s=1} = \frac{2}{h^+(x)}\sum_{n=0}^{N_T} a_n(x) n^2,$$

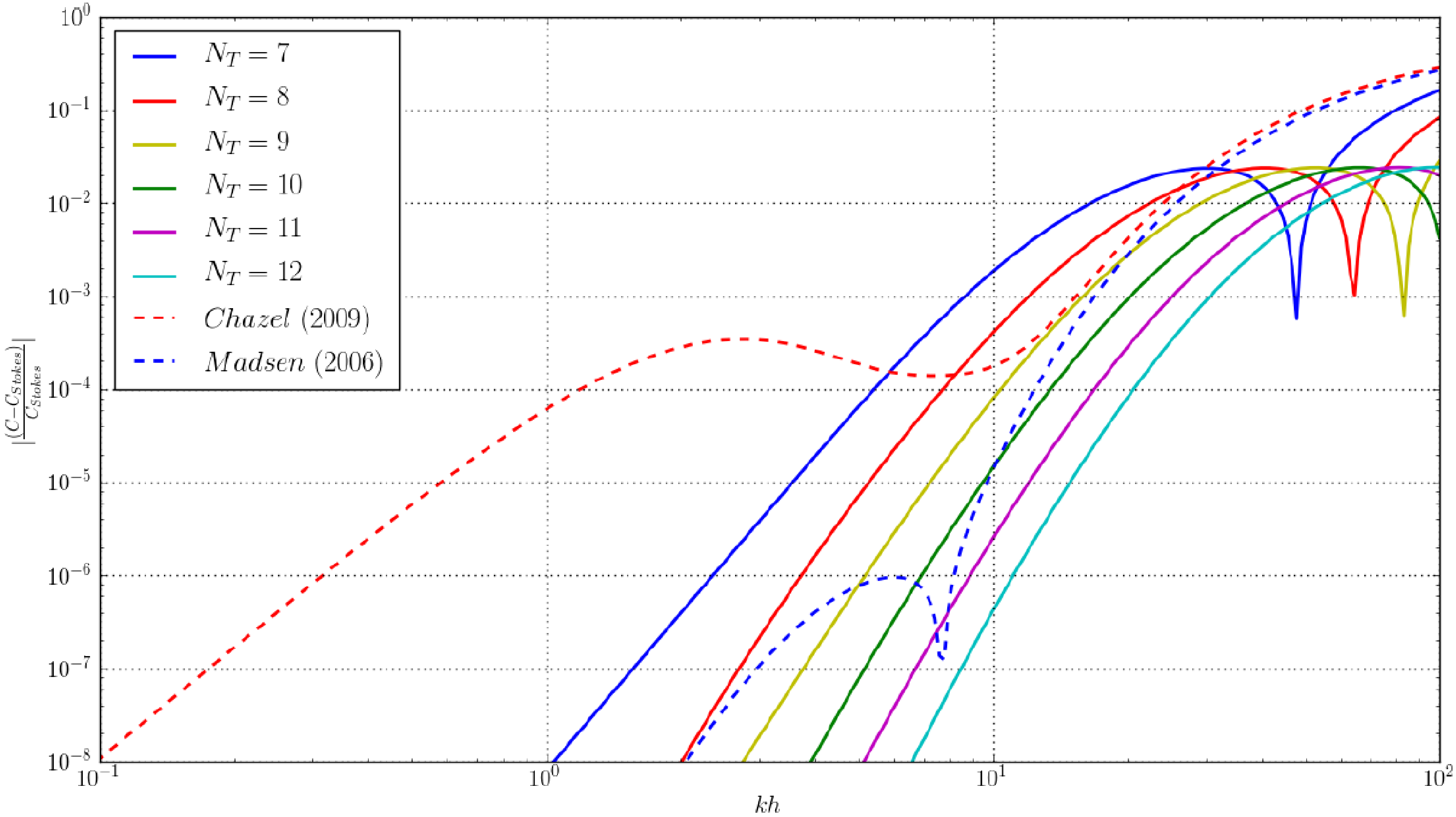
and the system can be stepped forward in time.

Dispersion relation of the LINEAR version of the model (flat bottom case)

Analytical expression $$\frac{\hat{\omega}^2_{N_T}}{\mu^2} = \left(\frac{C}{\sqrt{gh}}\right)^2_{N_T} = \frac{1+\sum_{p=1}^{N_T-2}\alpha_p\mu^{2p}}{1+\sum_{p=1}^{N_T-1}\beta_p\mu^{2p}}$$

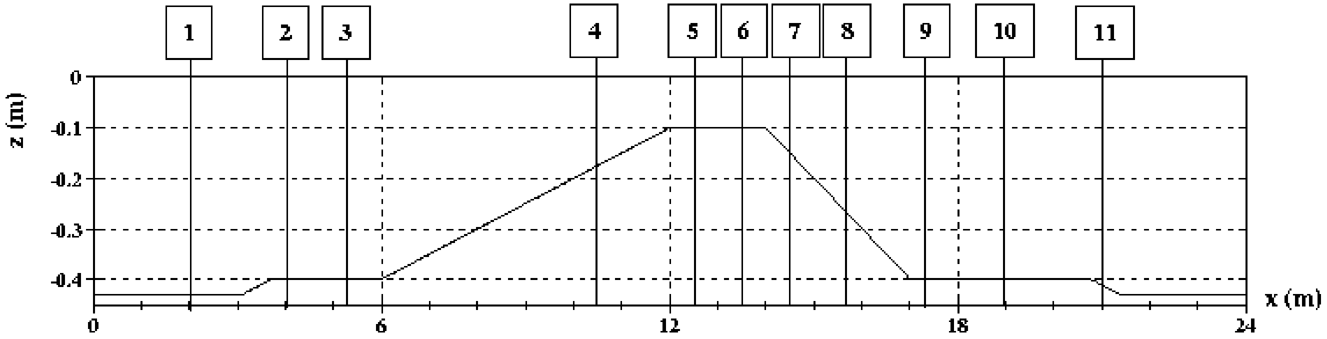
$$\mu = kh$$

$$\hat{\omega} \equiv \omega\sqrt{\frac{h}{g}}$$



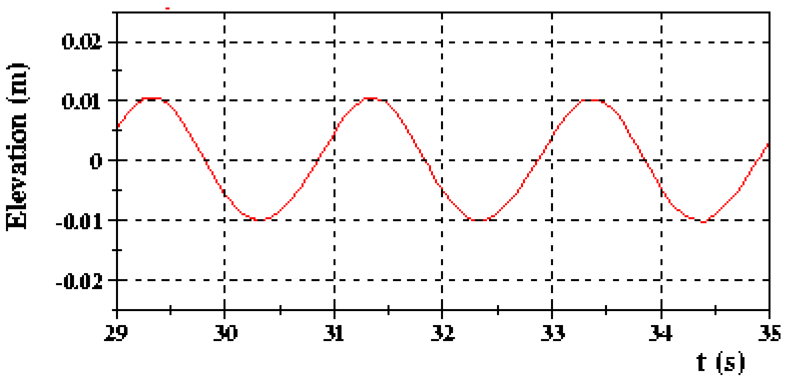
Regular non-breaking waves over a submerged bar (Dingemans exp.)

Experiments in wave flume by Beji & Battjes (1993), Luth *et al*. (1994), Dingemans (1994)
=> strong nonlinear (up to the top of the bar) and dispersive (after the bar) effects



Simulation of case A: T = 2.02 s, H = 2 cm

Wave signal at probe 2 =>

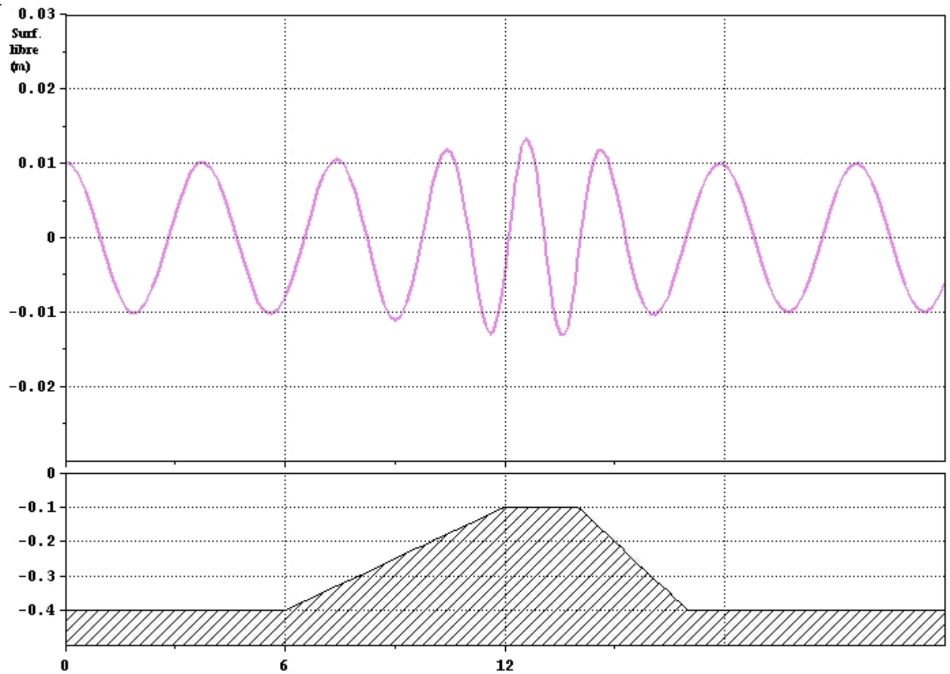


Setup and discretization:

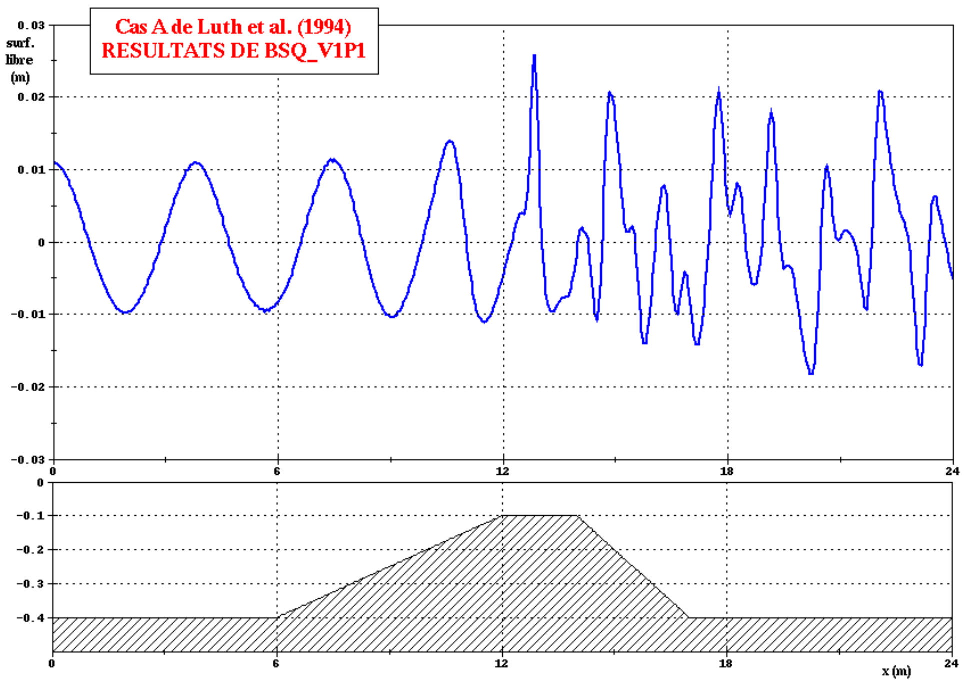
- linear wave solution at the left boundary.
- relaxation zones: left 7 m (generation) and right 7 m (absorption).
- horizontal axis: x = -6 to 38 m with constant Δx = 0.05 m => 881 nodes.
- order of Chebyshev polynomials: N_T = 3, 4, 5, 7, 10.
- time: t = 0 to 25T = 50.5 s with Δt = T/100 = 0.0202 s => 2 500 time steps.

Dingemans (1994) experiment A (T= 2.02 s H = 0.02 m)

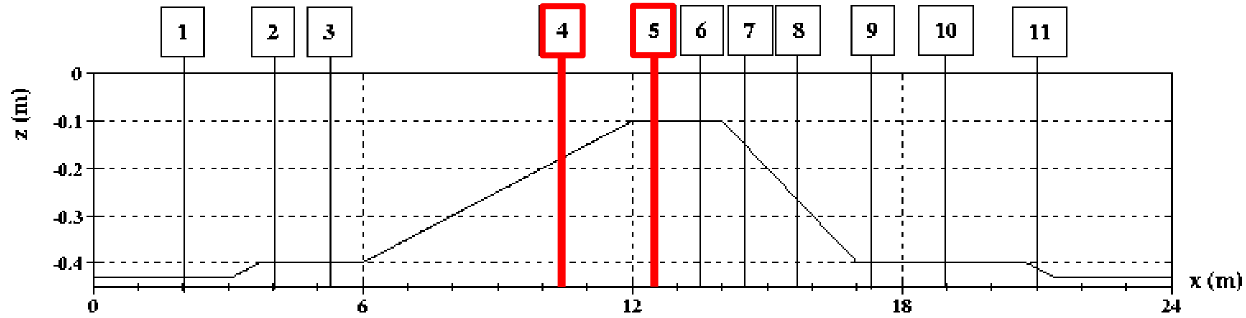
Linear dispersive model

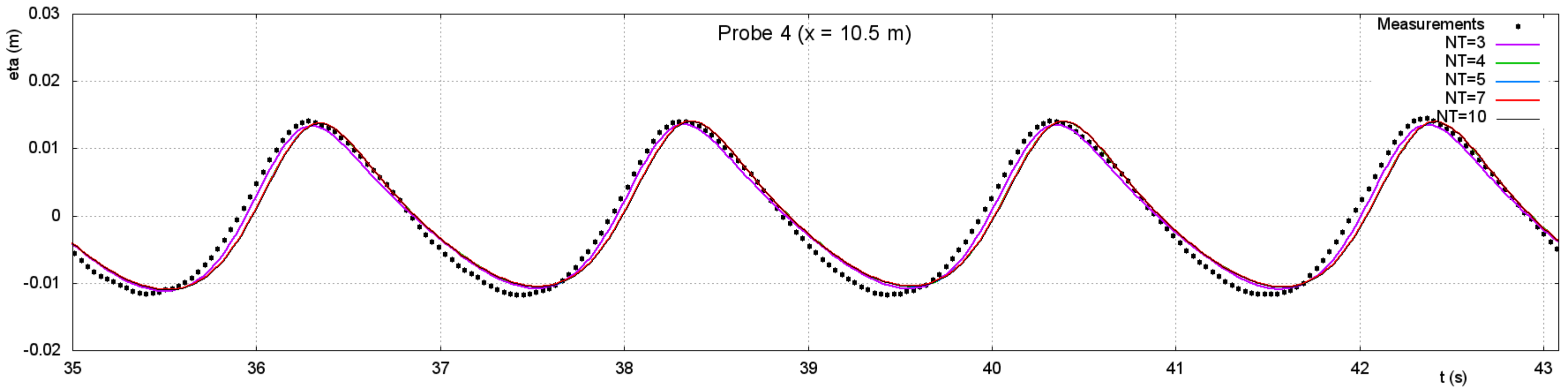


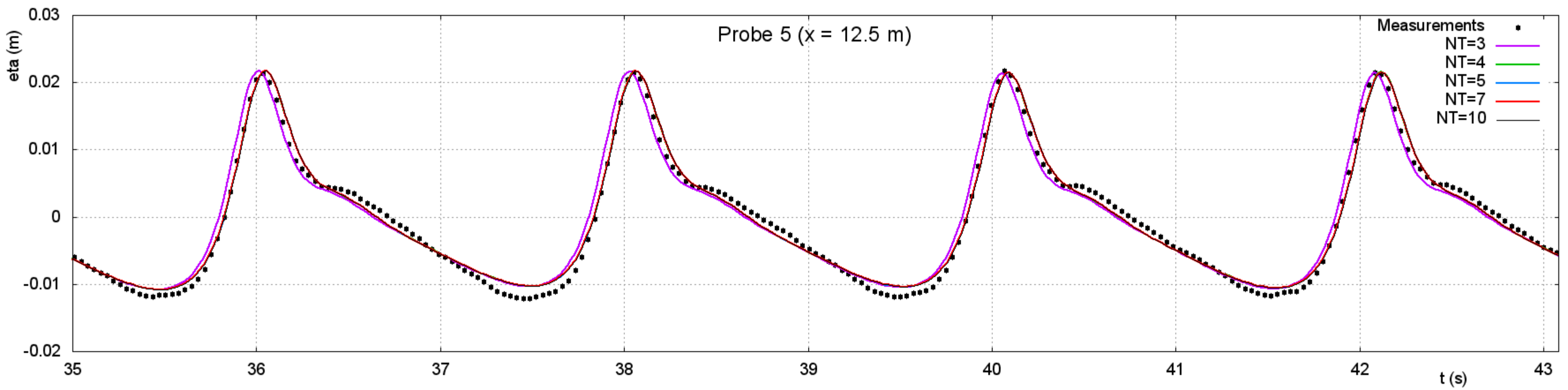
Nonlinear Boussinesq model



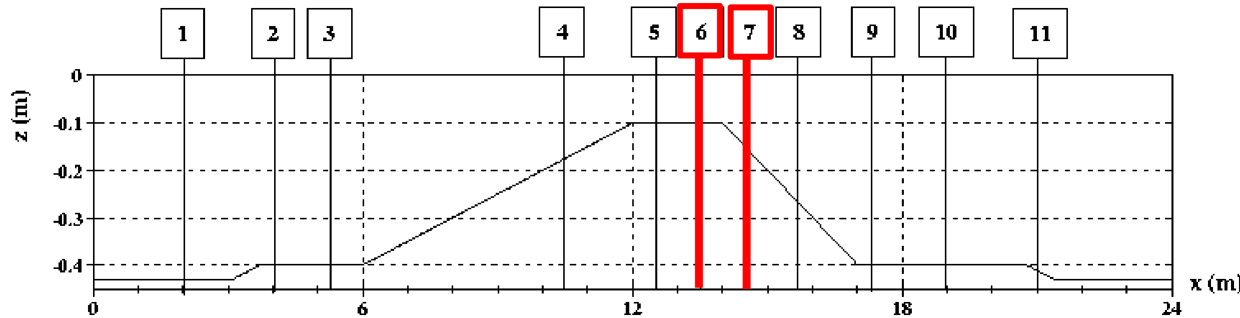
Results for case A
(T = 2.02 s, H = 2 cm)

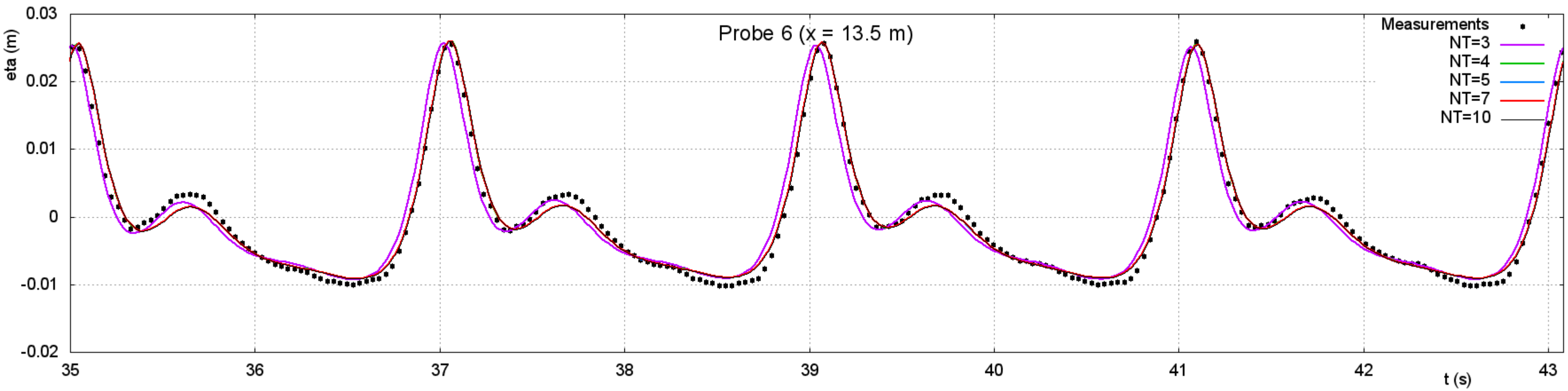


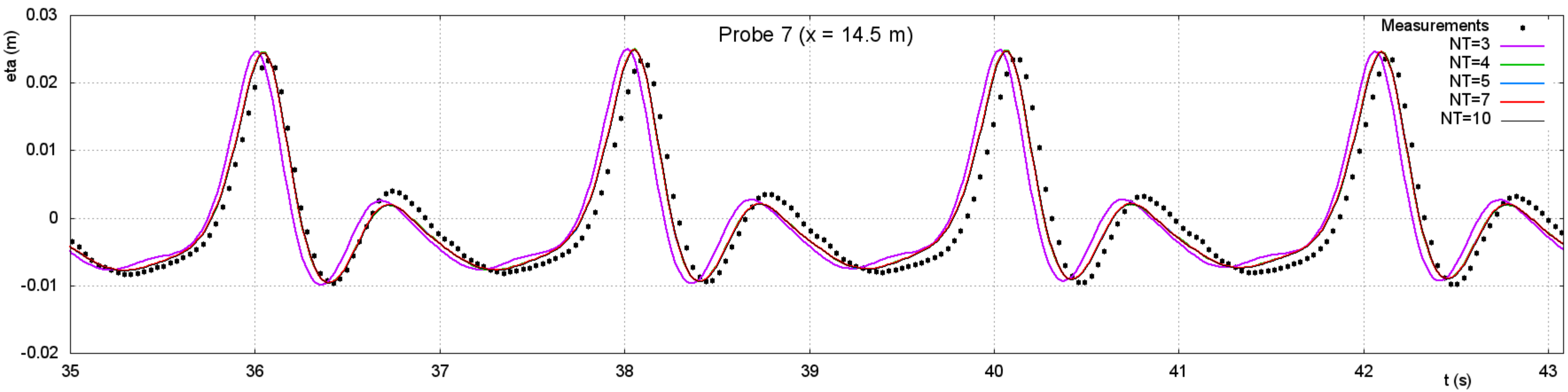




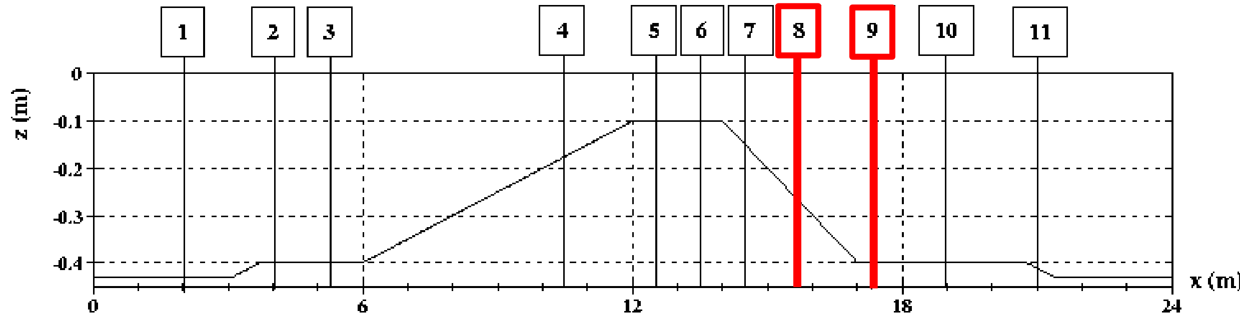
Results for case A
(T = 2.02 s, H = 2 cm)

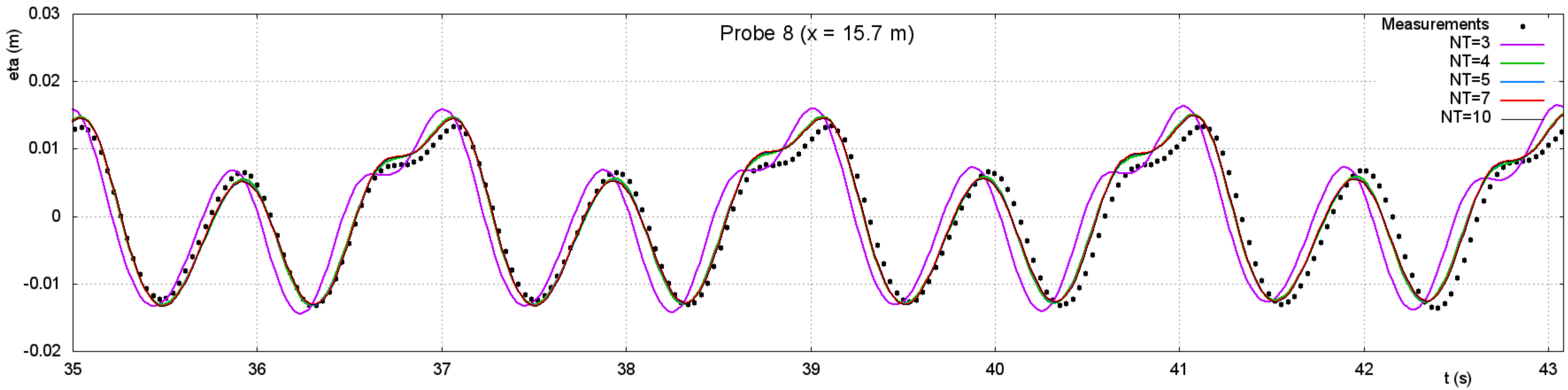


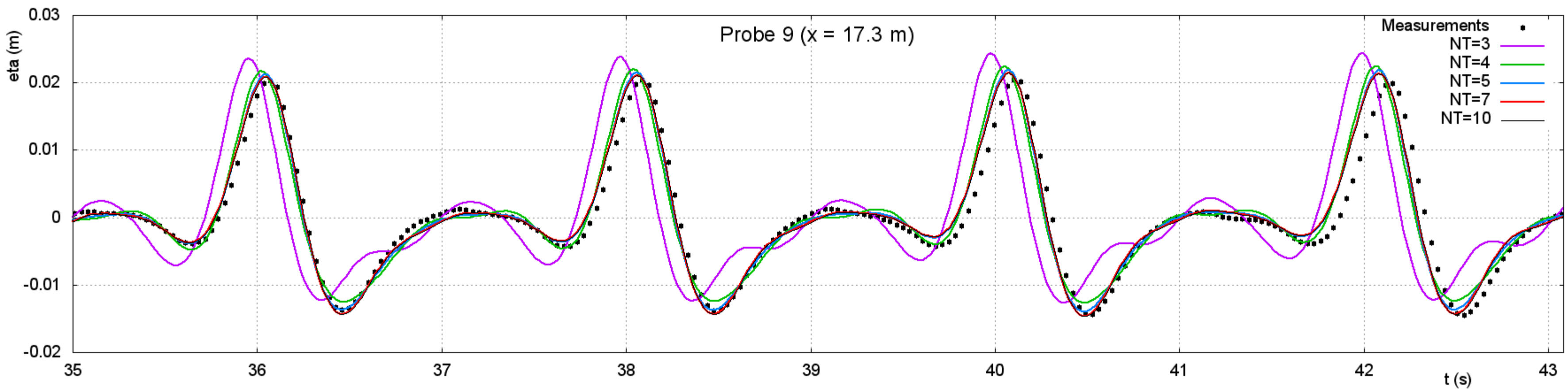




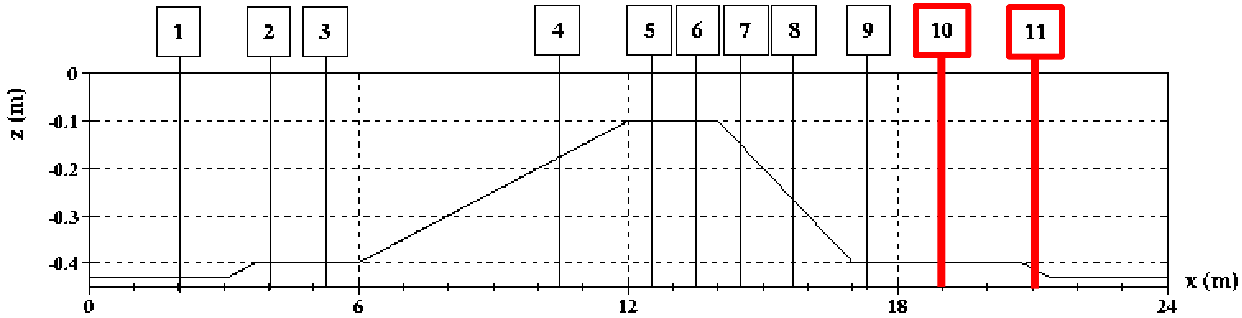
Results for case A
(T = 2.02 s, H = 2 cm)

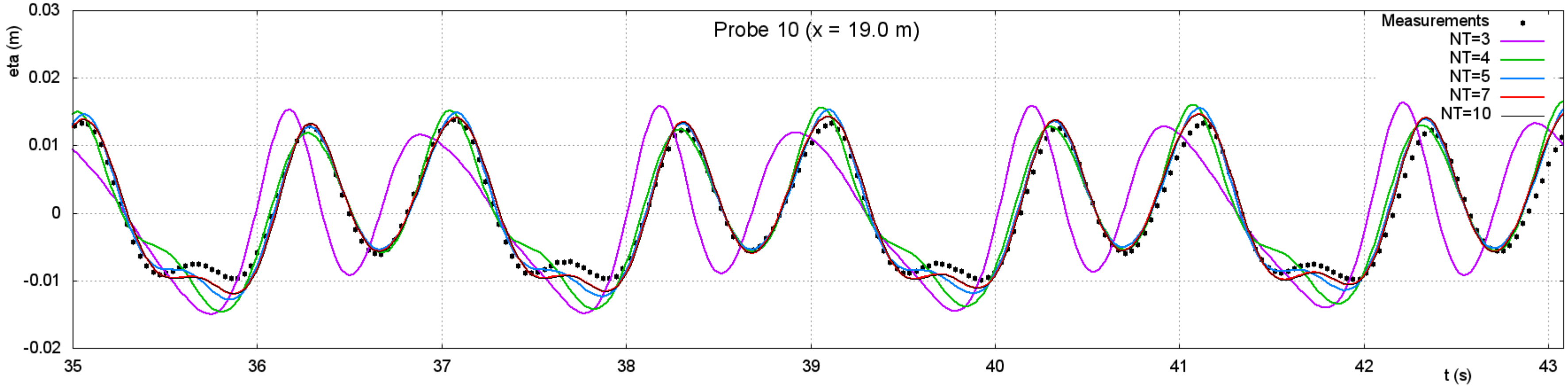


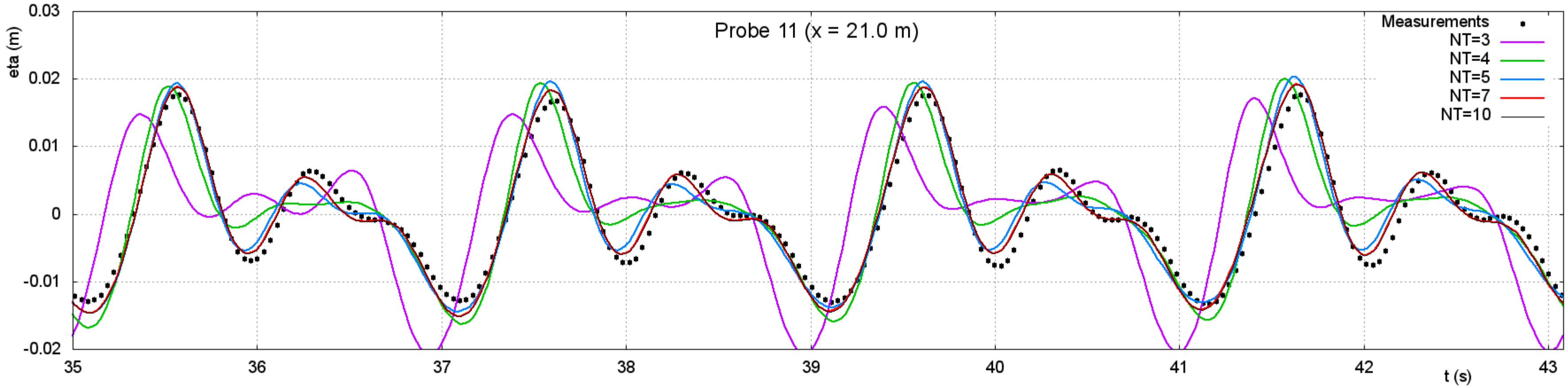




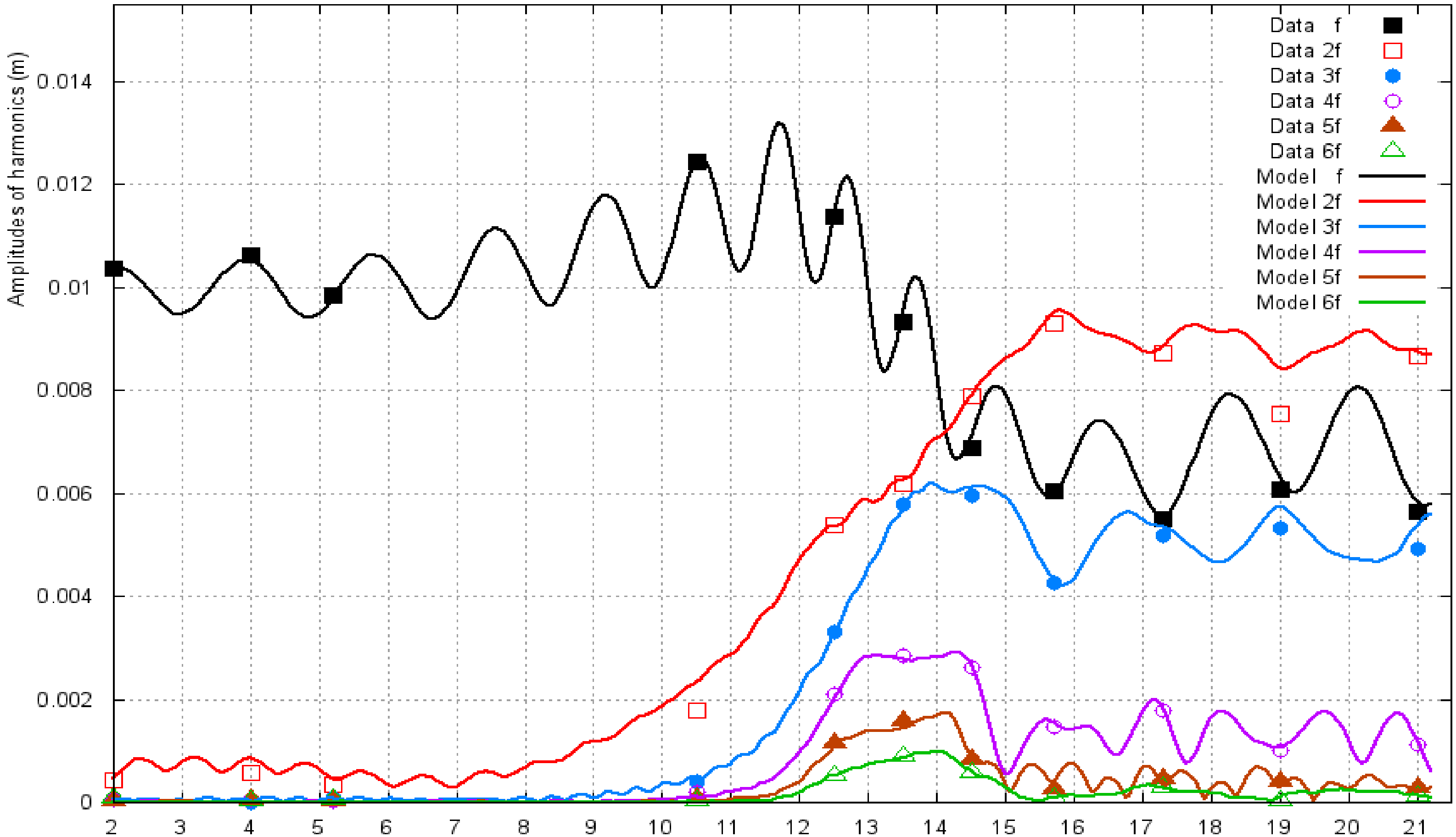
Results for case A
(T = 2.02 s, H = 2 cm)

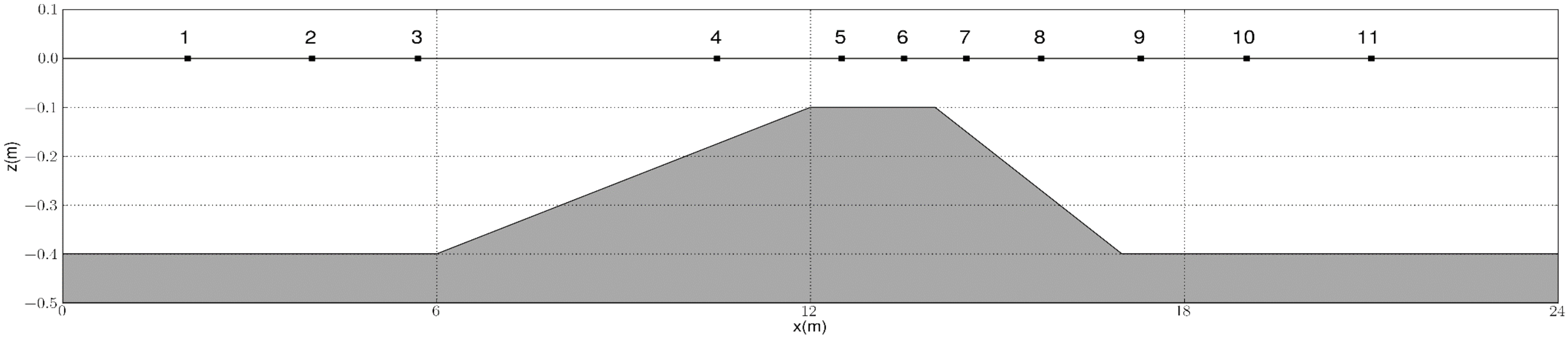






Evolution of amplitudes of the first six harmonics (Fourier analysis)





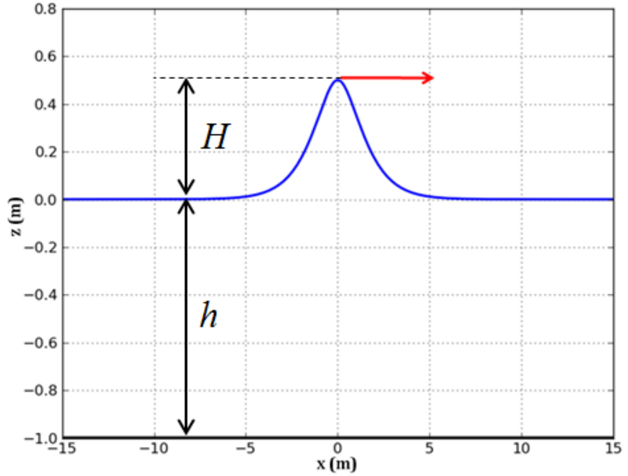
Part IV – On solitary waves

Experimental reproduction of the Scott Russell solitary wave in a water canal at Heriot-Watt Univ. in Scotland (12 July 1995).
Re-creation of the famous 1834 'first' sighting of a soliton or solitary wave by John Scott Russell on the Union Canal near Edinburgh.

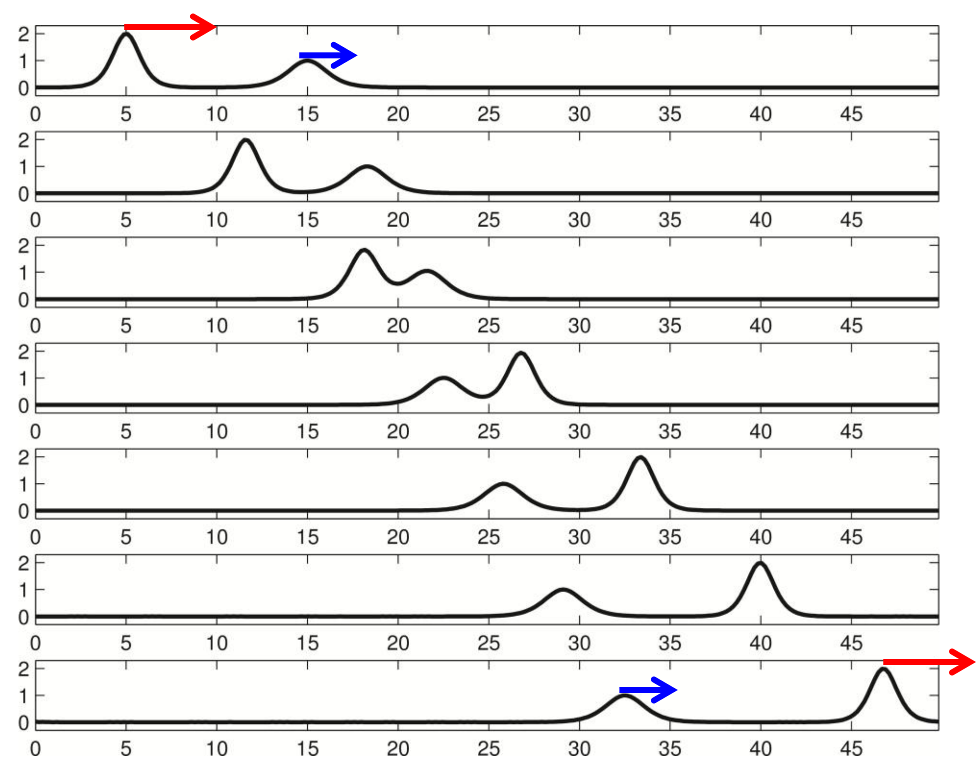
An introduction to solitary waves

Definition: a solitary wave is composed of a **single hump of water** which propagates with a **permanent stable shape** at a **constant speed** in a medium at rest at infinity.

The wave's speed depends on its height => **one single parameter, the ratio H/h.** (the larger the height, the faster the wave)



If two such waves collide they pass through each other and emerge from the collision unchanged.



Why considering solitary waves?

Historically extensively (and still frequently) used for representing tsunami waves (in theoretical studies, numerical and physical models)

…though it is now accepted that it is a poor model of real tsunami waves !

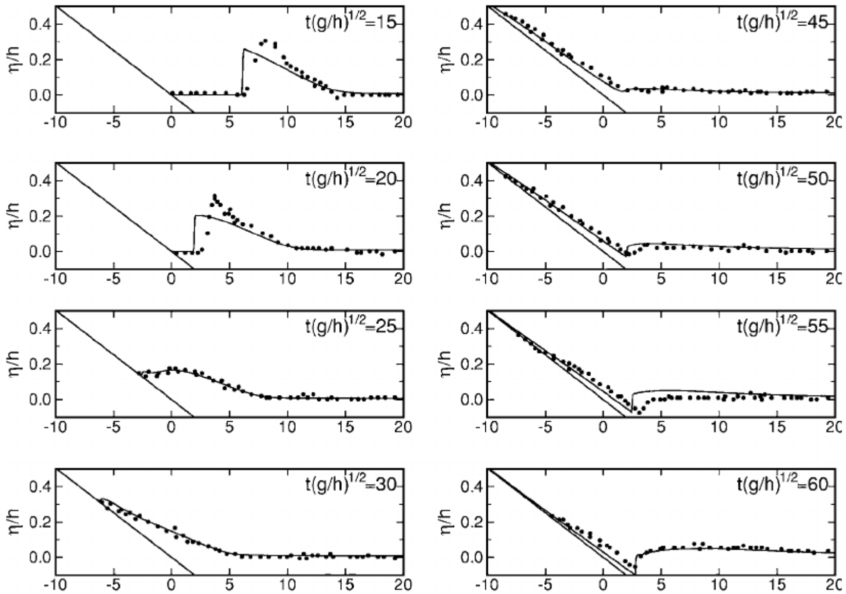
BUT...

it remains a good candidate to investigate the range of validity of mathematical models and the quality of numerical codes.

WHY?

Because solitary waves do exist as a result of balance between:

nonlinear effects (that tend to steepen the wave)
and
dispersive effect (that tend to spread and disperse the wave)**.**



Surface profiles of a solitary wave on a 1:19.85 plane beach from Synolakis' experiments (1987) for H/h = 0.3; and and comparison with a finite-volume model from Wei et al. (2006)

Why considering solitary waves?

Important remarks regarding solitary waves:

1. **Only models that include (some) dispersion and (some) nonlinear effects admit solitary wave solutions.**
   In particular, the NSWE model (Saint-Venant equations) does not admit a solitary wave solution. As no dispersive effects are present, any initial hump of water will steepen and evolve towards a discontinuity in a finite time.

2. **Different mathematical models exhibit different solitary wave profiles depending upon their dispersive and nonlinear properties.**
   There is no “one solitary wave”, although the Boussinesq solitary wave is often referred to as “the solitary wave solution”.

3. If a mathematical model admits a solitary wave solution, it is a good option to select this solution for a validation case of the numerical model implementing this mathematical model
   (but beware of using the proper theoretical solitary wave solution...).

Brief review of solitary waves solutions
(flat bottom – irrotational case)

Approximate mathematical models (compared to the full Euler equations):

$\Rightarrow$ Analytical expressions can be obtained for some integrable models such as:

- the KdV model
- <u>some</u> Boussinesq-type models (...but not all of them !)
- the Serre-Green-Naghdi model,

and in other cases a numerical method should be used to find approximate solutions.

$\Rightarrow$ Note that those formulas could be used whatever the ratio H/h set on input, ...which demonstrates they cannot be physical solutions for high amplitude waves !

Full Euler mathematical model:

No analytical closed expression exist.

- Approximate solutions:
  - Order 1 and 2 in $\varepsilon = H/h$ => Laitone (1959)
  - Order 3 in ε => Grimshaw (1971)
  - Order 9 in ε => Fenton (1972)
- Accurate numerical techniques/algorithms such as:
  - Tanaka (1986),
  - Clamond & Dutykh (2013); Dutykh & Clamond (2014).

=> Computations possible (but difficult) up to a maximum value of $H/h \approx 0.83$.

Notations

h = water depth (m)
H = height of the solitary wave (m)
C = celerity of the solitary wave (m/s)

Still water celerity of long waves: $C_0 = \sqrt{gh}$ (≠ C, a function of h only)

$X = x - Ct - x_0$ where x_0 is the initial position of the wave (m)

Some models (Boussinesq, KdV, etc.) use: K = pseudo wave-number (1/m)

Notations: $S = S(x,t) = \dfrac{1}{\cosh(KX)} = \operatorname{sech}(KX)$

$$T = T(x,t) = \tanh(KX)$$

We have: $S^2 = \dfrac{1}{\cosh^2(KX)} = \dfrac{2}{1+\cosh(2KX)} = 1 - T^2$

Characteristic numbers

Nondimensional numbers: **the solitary wave is function of a single parameter**

- relative wave height: $\varepsilon = H/h$: (0 to ~0.83 for Euler solitary wave)

- Froude number: $F = C/C_0 = C/\sqrt{gh}$ (1 to ~1.29421 for Euler solitary wave)

- nondimensional crest velocity : $q_c = |u_c|/\sqrt{gh}$ (0 to 1)

A solitary wave solution is defined by a relation between *H*, *h* and *C*,

or in non-dimensional form: $F^2 = \mathrm{f}(\varepsilon)$

Dispersion relation:

If the pseudo wave-number *K* is defined, the relative depth is: $\kappa = Kh$.

When *K* exists, a pseudo dispersion relation can be written: $F^2 = \mathrm{g}(\kappa)$

Solitary wave of the KdV equation (Korteweg & De Vries, 1895)

Exact (analytic) solution of the KdV equation (right-going waves only):

$$\eta_t + C_0\eta_x + \frac{3}{2}\frac{C_0}{h}\eta\eta_x + \frac{1}{6}C_0 h^2 \eta_{xxx} = 0$$

Free surface: $\eta(x,t) = H\,\text{sech}^2\left(K(x - Ct - x_0)\right) = H\,\text{sech}^2\left(KX\right)$

Celerity: $C = \sqrt{gh}\left(1 + \dfrac{H}{2h}\right)$ or $F^2 = \left(1 + \dfrac{\varepsilon}{2}\right)^2 \approx 1 + \varepsilon \quad \text{if} \quad \varepsilon << 1$

Pseudo wave-number: $K = \sqrt{\dfrac{3H}{4h^3}}$ or $\kappa = \sqrt{\dfrac{3}{4}\varepsilon}$

Dispersion relation: $F = 1 + \dfrac{2}{3}\kappa^2$ $\qquad F^2 = \left(1 + \dfrac{2}{3}\kappa^2\right)^2 \approx 1 + \dfrac{4}{3}\kappa^2 \quad \text{if} \quad \kappa << 1$

References:

Korteweg D.J., De Vries G. (1895) On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave. Phil. Mag., Vol. 39(5), pp 422-443.

Solitary wave of the Boussinesq single-equation model

Exact solution of the Boussinesq single-equation (right-going waves only):

$$\eta_{tt} + C_0^2 \eta_{xx} + \frac{3}{2}\frac{C_0^2}{h}\left(\eta^2\right)_{xx} - \frac{C_0^2 h^2}{3}\eta_{xxxx} = 0$$

Free surface: $\eta(x,t) = H \operatorname{sech}^2\left(K(x - Ct - x_0)\right)$

Celerity: $C = \sqrt{gh\left(1 + \frac{H}{h}\right)} = \sqrt{g\left(h+H\right)}$ or $F = \sqrt{1+\varepsilon}$ or $F^2 = 1+\varepsilon$

Note that $\sqrt{1+\frac{H}{h}} < 1 + \frac{1}{2}\frac{H}{h}$ and so $C_{Bous1} < C_{KdV}$

Pseudo wave-number: $K = \sqrt{\frac{3H}{4h^3}}$ or $\kappa = \sqrt{\frac{3}{4}\varepsilon}$

Dispersion relation: $F^2 = 1 + \frac{4}{3}\kappa^2$

References:

Boussinesq J.V. (1871) Théorie de l'intumescence liquide, appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire. C. R. Acad. Sci. Paris, Vol. 72, pp 755–759.

Solitary wave of the Serre-Green-Naghdi (SGN) equations

Exact solution of the SGN (or Su-Gardner) equations (for constant water depth h here):

$$\eta_t + \left((h+\eta)U\right)_x = 0$$

$$U_t + UU_x + g\eta_x = \frac{1}{3(h+\eta)}\left[(h+\eta)^3\left(U_{xt} + UU_{xx} - U_x^2\right)\right]_x$$

Free surface: $\eta(x,t) = H\,\mathrm{sech}^2(KX)$

Velocity (from the above continuity equation): $U(x,t) = \dfrac{C\,\eta(x,t)}{h+\eta(x,t)}$

Celerity: $C = \sqrt{gh\left(1+\dfrac{H}{h}\right)}$ or $F = \sqrt{1+\varepsilon}$ or $F^2 = 1+\varepsilon$

Pseudo wave-number: $K = \sqrt{\dfrac{3H}{4h^2(h+H)}}$ or $\kappa = \sqrt{\dfrac{3}{4}\dfrac{\varepsilon}{1+\varepsilon}}$

Dispersion relation: $F^2 = \dfrac{1}{1-\frac{4}{3}\kappa^2} \approx 1 + \dfrac{4}{3}\kappa^2 \quad \text{if} \quad \kappa << 1$

References:

Solitary wave of Euler eq. at orders 1 & 2 (Laitone, 1959)

Approximate solutions at order 1 and 2 in ε of the full Euler equations:

Order 1:
(idem KdV)

$$\eta(x,t) = H\,\mathrm{sech}^2(KX) \qquad C = \sqrt{gh}\left(1+\frac{H}{2h}\right) \qquad K = \sqrt{\frac{3H}{4h^3}}$$

Order 2:

$$\eta(x,t) = H\,\mathrm{sech}^2(KX) - \frac{3}{4}\frac{H^2}{h}\mathrm{sech}^2(KX)\left[1-\mathrm{sech}^2(KX)\right] \qquad X = x - Ct - x_0$$

Pseudo wave-number:

$$K = \sqrt{\frac{3H}{4h^3}}\left(1-\frac{5}{8}\frac{H}{h}\right) \qquad \kappa = \sqrt{\frac{3}{4}\varepsilon}\left(1-\frac{5}{8}\varepsilon\right)$$

Dispersion relation:

$$C = \sqrt{gh}\left(1+\frac{1}{2}\frac{H}{h}-\frac{3}{20}\left(\frac{H}{h}\right)^2\right)$$

$$F^2 = \left(1+\frac{1}{2}\varepsilon-\frac{3}{20}\varepsilon^2\right)^2 \approx 1+\varepsilon-\frac{1}{20}\varepsilon^2-\frac{3}{20}\varepsilon^3 \quad \text{if} \quad \varepsilon << 1$$

References:

Laitone E. (1959) Water Waves, IV: Shallow Water Waves, University of California, Berkeley, Institute of Engineering Research Technical Report 82-11.

Zhao B.B., Ertekin R.C., Duan W.Y., Hayatdavoodi M. (2014) On the steady solitary-wave solution of the Green–Naghdi equations of different levels. Wave Motion, Vol. 51, pp 1382-1395.

Solitary wave of Euler eq. at order 3 (Grimshaw, 1971)

Approximate solutions at order 3 in ε of the full Euler equations (in Wu et al., 2014) :

Free surface:

$$\eta = h\left(\left(\frac{H}{h}\right)S^2 - \frac{3}{4}\left(\frac{H}{h}\right)^2 S^2T^2 + \left(\frac{H}{h}\right)^3\left(\frac{5}{8}S^2T^2 - \frac{101}{80}S^4T^4\right)\right)$$

Celerity:

$$C = \sqrt{gh}\left(1 + \frac{1}{2}\left(\frac{H}{h}\right) - \frac{3}{20}\left(\frac{H}{h}\right)^2 + \frac{3}{56}\left(\frac{H}{h}\right)^3\right)$$

or

$$F^2 = \left(1 + \frac{1}{2}\varepsilon - \frac{3}{20}\varepsilon^2 + \frac{3}{56}\varepsilon^3\right)^2 \approx 1 + \varepsilon - \frac{1}{20}\varepsilon^2 - \frac{3}{70}\varepsilon^3 \quad \text{if} \quad \varepsilon << 1$$

Pseudo wave-number:

$$K = \sqrt{\frac{3H}{4h^3}}\left(1 - \frac{5}{8}\left(\frac{H}{h}\right) + \frac{71}{128}\left(\frac{H}{h}\right)^2\right)$$

or

$$\kappa = \sqrt{\frac{3}{4}\varepsilon}\left(1 - \frac{5}{8}\varepsilon + \frac{71}{128}\varepsilon^2\right)$$

Fenton (1972) gives

$$\eta = 1 + \epsilon s^2 - \tfrac{3}{4}\epsilon^2 s^2 t^2 + \epsilon^3(\tfrac{5}{8}s^2t^2 - \tfrac{101}{80}s^4t^2) + O(\epsilon^4), \tag{20a}$$

$$F^2 = 1 + \epsilon - \tfrac{1}{20}\epsilon^2 - \tfrac{3}{70}\epsilon^3 + O(\epsilon^4), \tag{20b}$$

$$\alpha = (\tfrac{3}{4}\epsilon)^{\frac{1}{2}}(1 - \tfrac{5}{8}\epsilon + \tfrac{71}{128}\epsilon^2) + O(\epsilon^{\frac{7}{2}}), \tag{20c}$$

References:

Grimshaw R.H.J. (1971) The solitary wave in water of variable depth, part 2. J. Fluid Mech., Vol. 46, pp 611–622.

Wu. N.-J., Tsay T.-K., Chen Y.-Y. (2014) Generation of stable solitary waves by a piston-type wave maker. Wave Motion, Vol. 51, pp 240-255.

Solitary wave of Euler eq. at order 9 (Fenton, 1972)

Approximate solutions at order 9 in ε of the full Euler equations :

Free surface: $$\eta = h\left(\left(\frac{H}{h}\right)\eta_1 + \left(\frac{H}{h}\right)^2 \eta_2 + \cdots + \left(\frac{H}{h}\right)^9 \eta_9\right)$$

Celerity: $$c^2 = gh\left(1 + \left(\frac{H}{h}\right)C_1 + \left(\frac{H}{h}\right)^2 C_2 + \cdots + \left(\frac{H}{h}\right)^9 C_9\right)$$

or $$F^2 = 1 + C_1\varepsilon + C_2\varepsilon^2 + \ldots + C_9\varepsilon^9$$

Pseudo wave-number: $$K = \sqrt{\frac{3H}{4h^3}}\left(1 + \left(\frac{H}{h}\right)K_1 + \left(\frac{H}{h}\right)^2 K_2 + \cdots \left(\frac{H}{h}\right)^9 K_9\right)$$

or $$\kappa = \sqrt{\frac{3}{4}\varepsilon}\left(1 + K_1\varepsilon + K_2\varepsilon^2 + \ldots + K_9\varepsilon^9\right)$$

Coefficients C_i and K_i are given in Table 1 of Fenton (1972)

References :
Fenton J. (1972) A ninth-order solution for the solitary wave. J. Fluid Mech., Vol. 52, pp 257–271

Solitary wave profiles – some comparisons

***H/h* = 0.556**

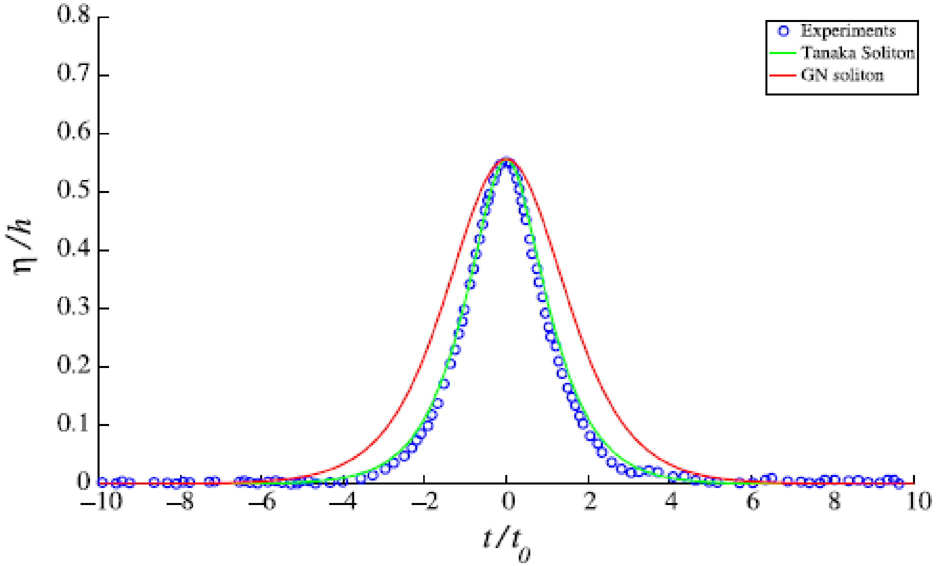


Fig. 2. Experimental solitary wave (○), Tanaka solitary wave (green line) and Green–Naghdi solitary wave (red line) for $a/h = 0.556$ and $\tau = \sqrt{h/g}$.

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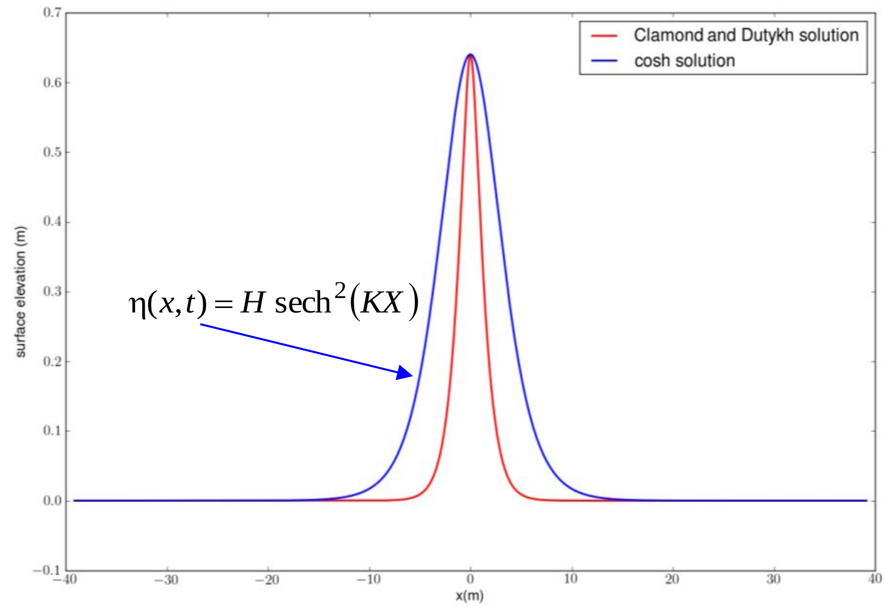
An experimental study of steep solitary wave reflection at a vertical wall

Y.Y. Chen [a], C. Kharif [b,*], J.H. Yang [a], H.C. Hsu [a], J. Touboul [c,d], J. Chambarel [b]

***H/h* = 0.64**

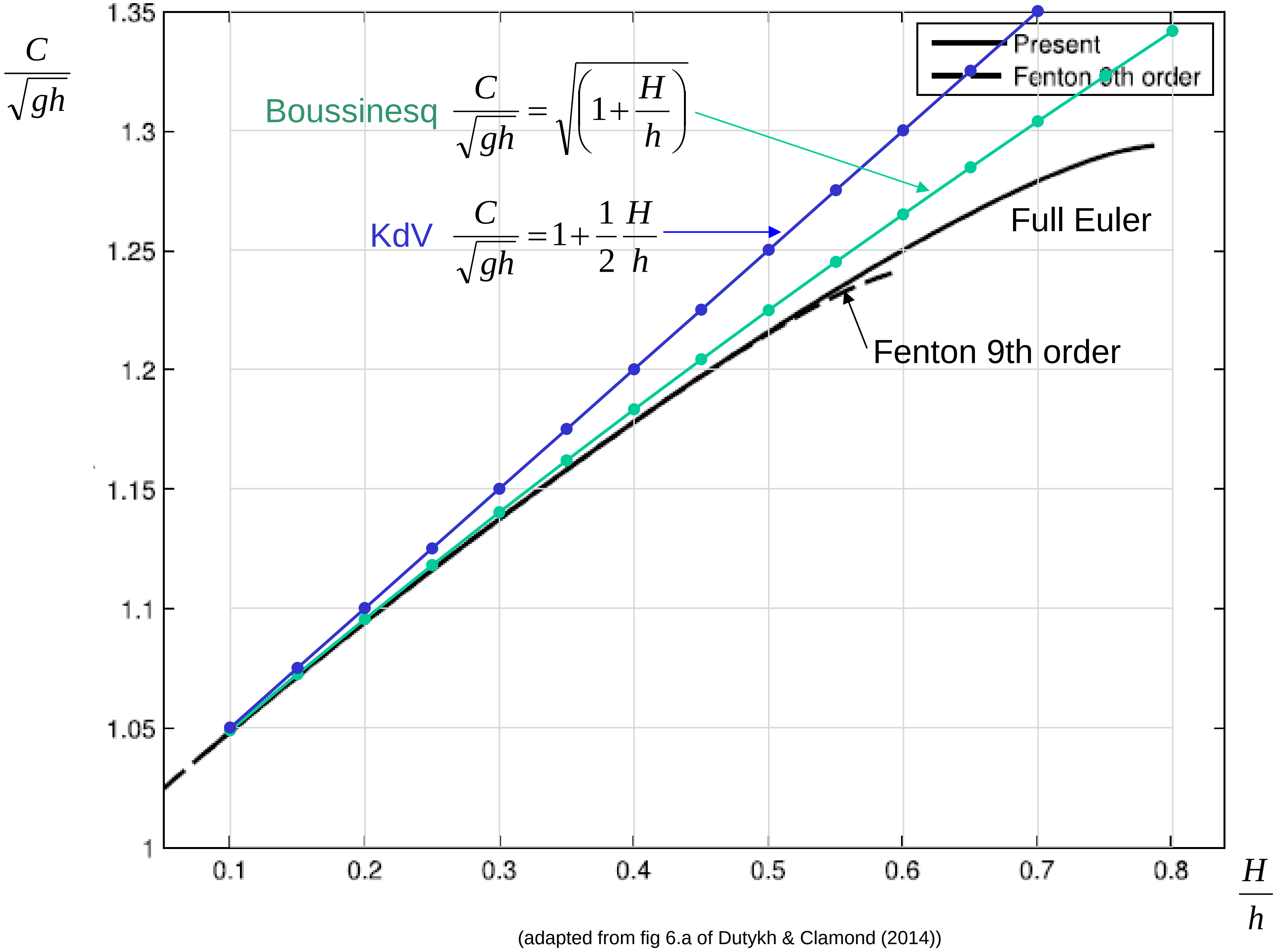


Comparison of the Boussinesq solitary wave solution (blue) and the full-Euler solution (red) by Clamond & Dutykh (2013) algorithm for *H/h* = 0.64.

Full Euler solitary solutions are more narrow than Boussinesq or KdV solitary solutions.

Differences more marked as H/h increases.

Solitary wave celerity



(adapted from fig 6.a of Dutykh & Clamond (2014))

TANDEM project - Work-package WP1

P01 - Solitary wave of Euler equation – long-distance propagation

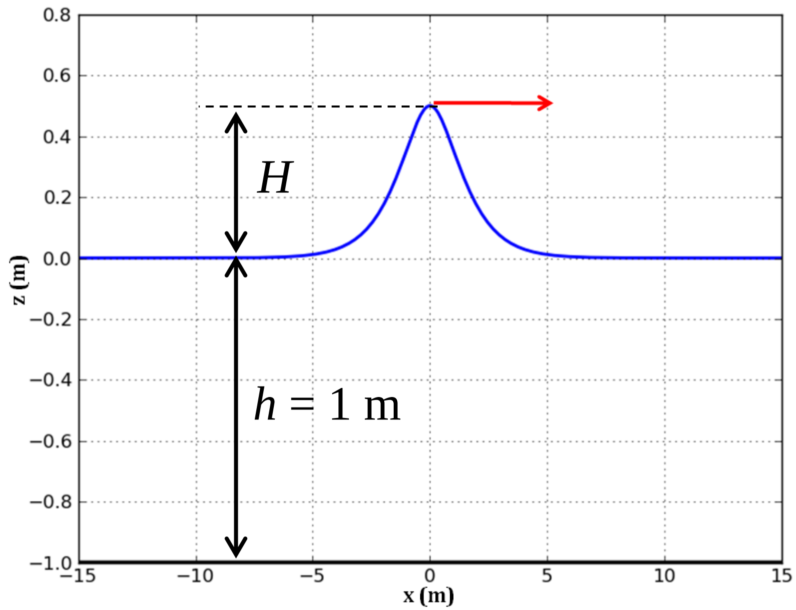
Goal of this test case:

- propagate a solitary wave over a flat bottom with minimum distortion or phase difference, even over long distances,
- reference solution available for a primitive model (full Euler equations),
- evaluate the accuracy and capabilities of numerical codes for propagating such waves over long distances.

TANDEM test-case P01 - Definition of test-case

Solitary wave height solution of the **full Euler equations** on flat bottom (no assumption on nonlinearity nor on dispersion).

Initial conditions and reference solution: highly accurate numerical method and algorithm by Clamond & Dutykh (2013).



Wave height: $\delta = H/h = 0.3$, **0.5** or 0.7

Spatial domain x/h from -25 to 675 (total length = $700h$)

Propagation over long time $\tilde{T} \equiv T\sqrt{g/h} = 500$

Travelled distance for the case $\delta = 0.5$: $d \approx 608h$

P01 –H/h = 0.5 – Dicretisations and parameters

Five codes, based on different mathematical models.

Code	$M_x = 5$	$M_x = 10$	$M_x = 20$	$M_x = 40$	Comments
SLOWS (G-N) (INRIA)	$M_t = 25$	$M_t = 50$	$M_t = 100$	$M_t = 200$	**CFL = 0.20** ($M_t = 5\ M_x$) in all cases
TUCWave (Nwogu) (INRIA)	$M_t = 25$	$M_t = 50$	$M_t = 100$	$M_t = 200$	**CFL = 0.20** ($M_t = 5\ M_x$) in all cases
Telemac-3D (St-Venant Lab.)	$M_t \approx 31.6$ 5 levels	$M_t \approx 63.3$ 5 levels	$M_t \approx 126.6$ 10 levels	$M_t \approx 253.1$ 20 levels	**CFL ≈ 0.158** ($M_t \approx 6.3\ M_x$) in all cases
MISTHYC (IRPHE & St-Venant Lab.)	$M_t = 4$ $N_T = 7$	$M_t = 8$ $N_T = 7$	$M_t = 16$ $N_T = 7$	$M_t = 32$ $N_T = 7$	**CFL = 1.25** ($M_t = 0.8\ M_x$) in all cases
CLIONA (CEA)	$M_t = 7$	$M_t = 13$	$M_t = 27$	$M_t = 53$	**CFL ≈ 0.75** ($M_t \approx 1.33\ M_x$) in all cases

$\Delta x/h \equiv 1/M_x$ ➔ Nb of cells along $x = 700.M_x$

$\Delta\tilde{t} \equiv \Delta t\sqrt{g/h} = 1/M_t$ ➔ Nb of time steps = $500.M_t$

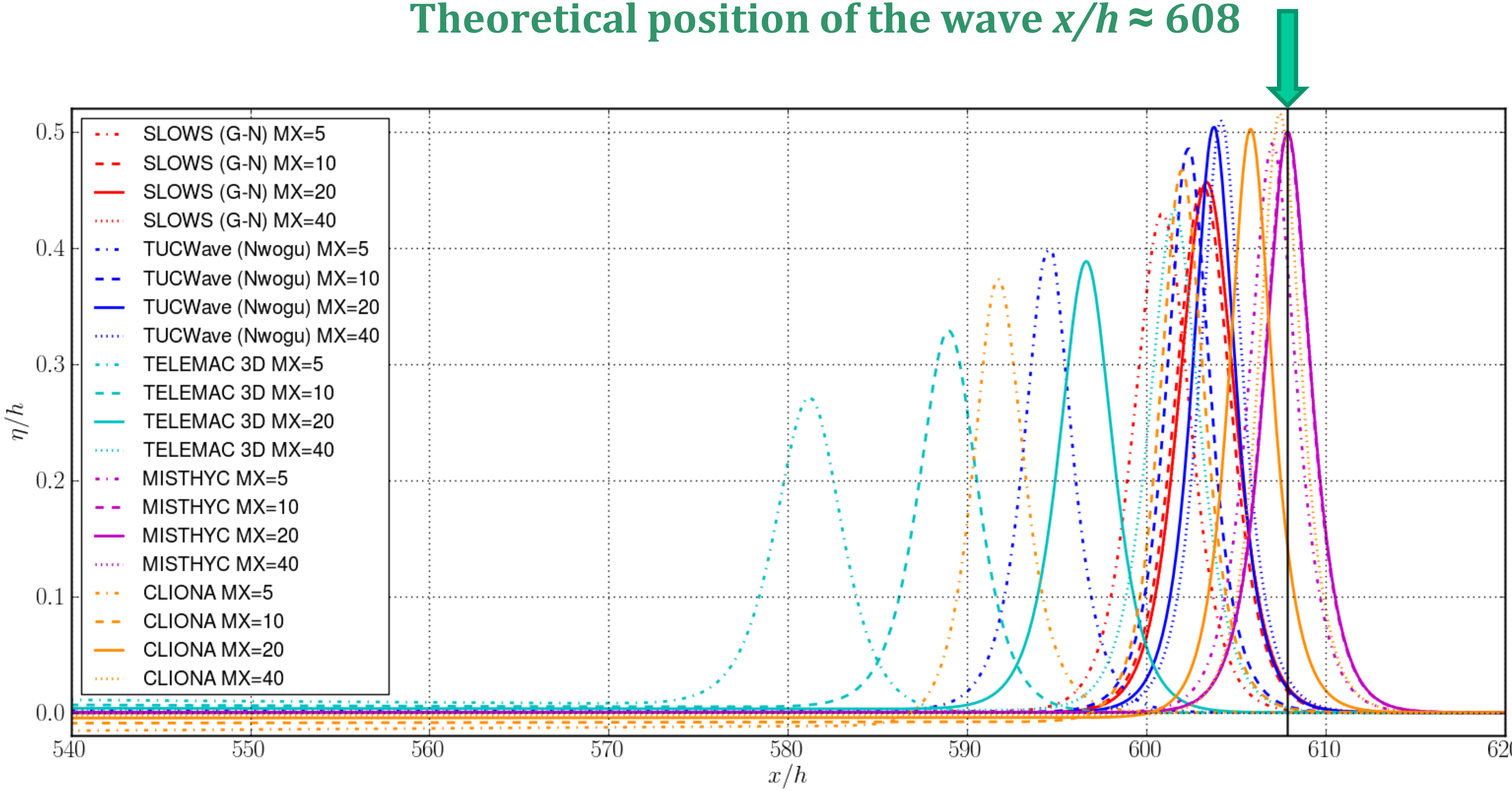
$$\mathrm{CFL} \equiv C_0 \Delta t/\Delta x = M_x/M_t$$

$$C_0 \equiv \sqrt{gh}$$

P01 – *H/h* = 0.5 – Free surface at final time $\tilde{T} \equiv T\sqrt{g/h} = 500$

(All results - zoom in the range *x/h* = 540 to 620)

Theoretical position of the wave *x/h* ≈ 608

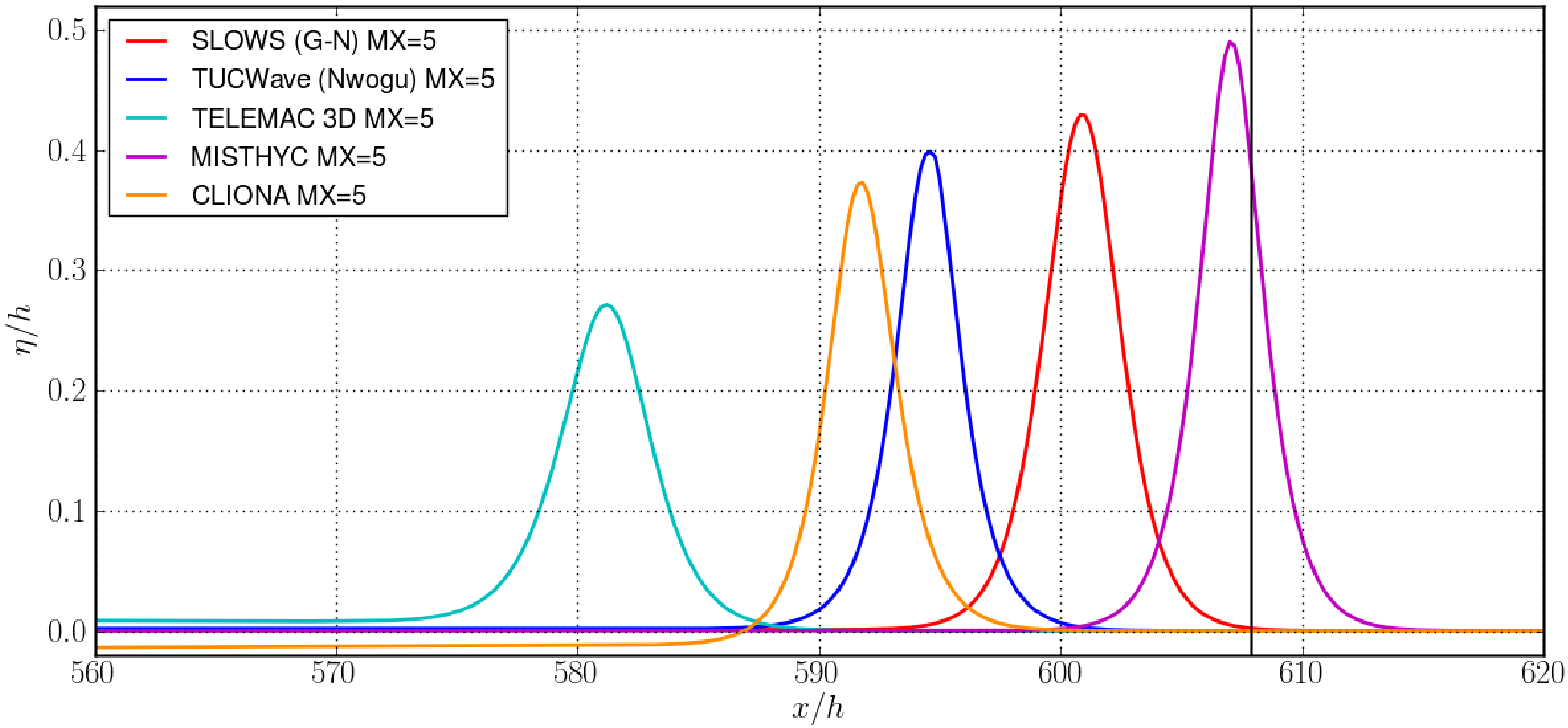


H/h = 0.5 – Free surface at final time

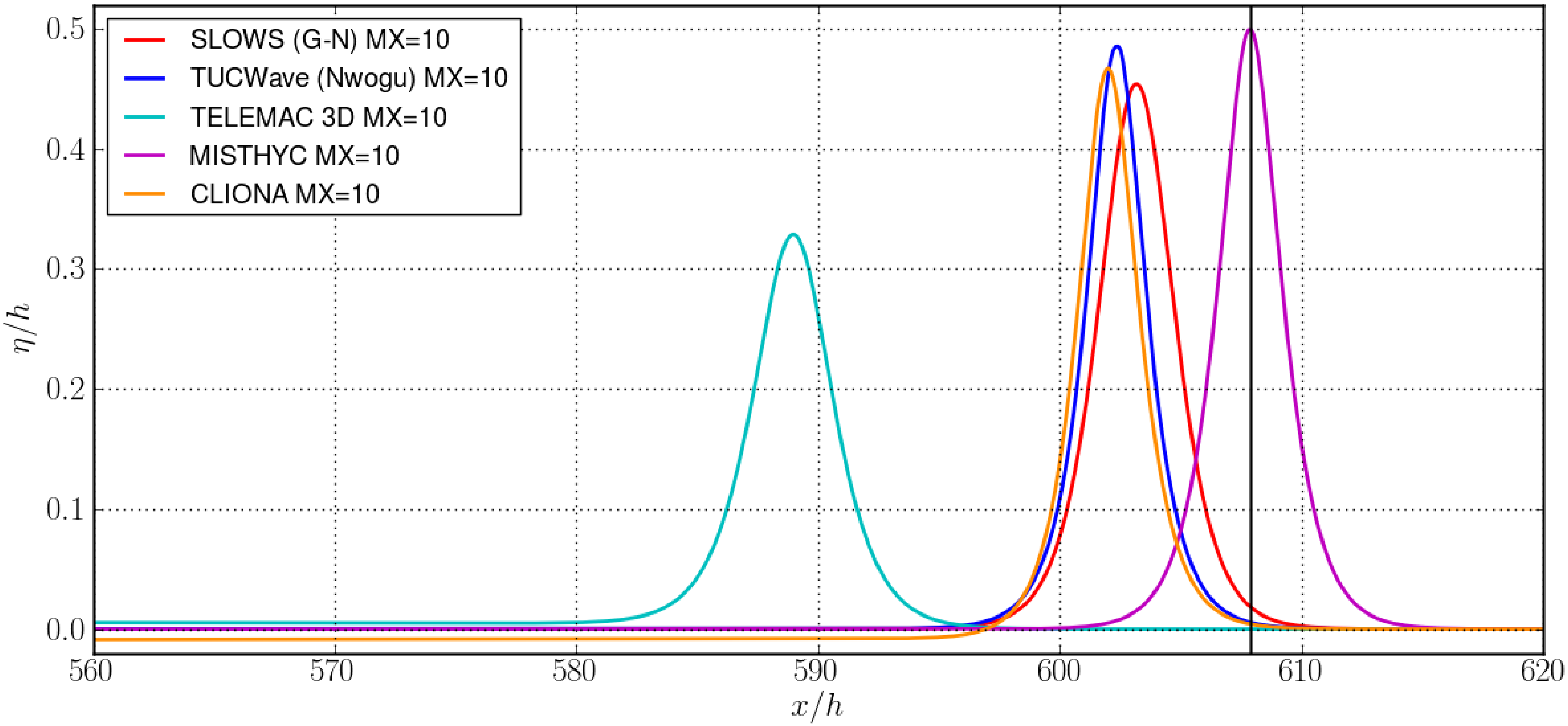
Comparison of all codes for each spatial resolution ($\Delta x \equiv h/M_x$)

Zoom in the range x/h = 560 to 620

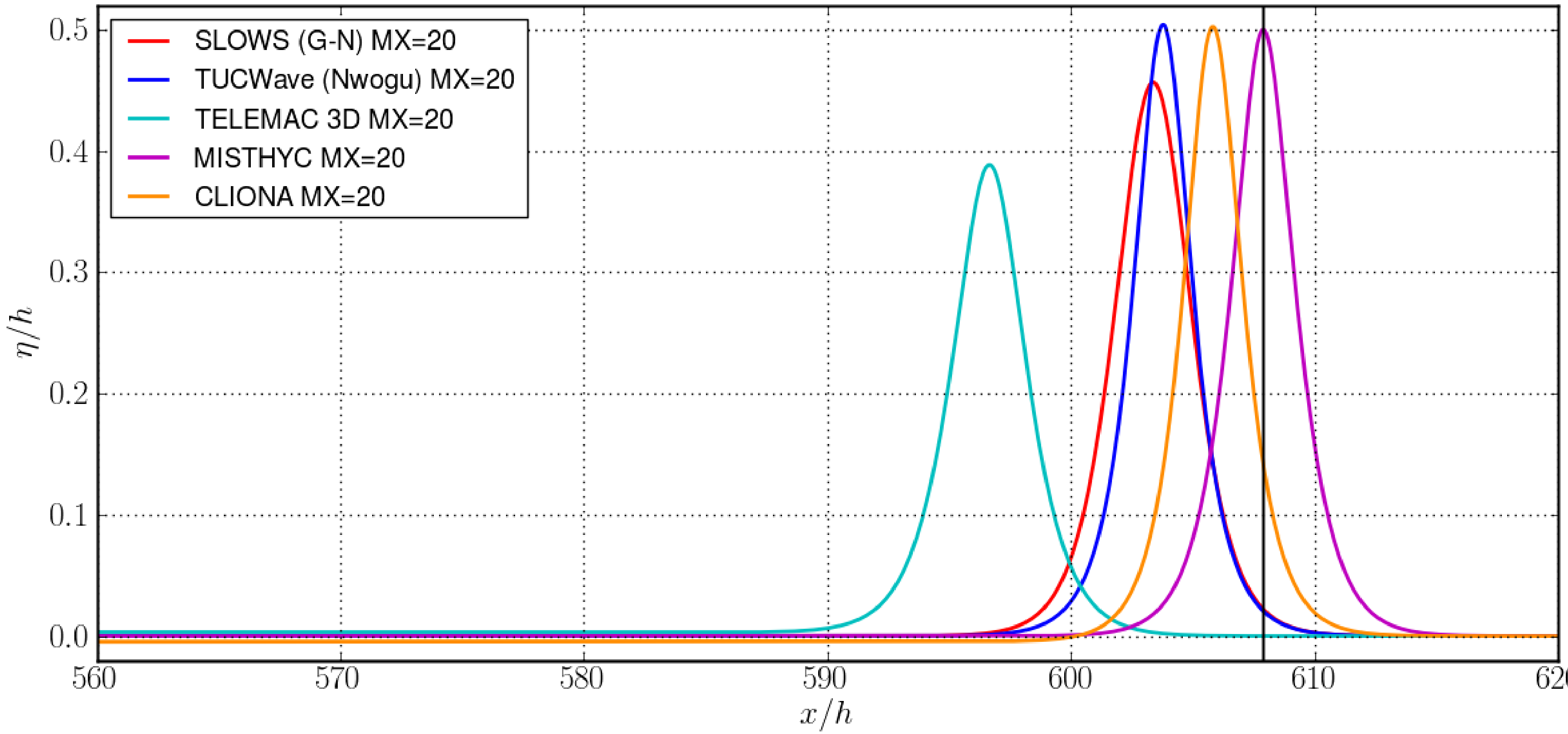
$H/h = 0.5$ – Free surface at final time – $M_x = h/\Delta x = 5$



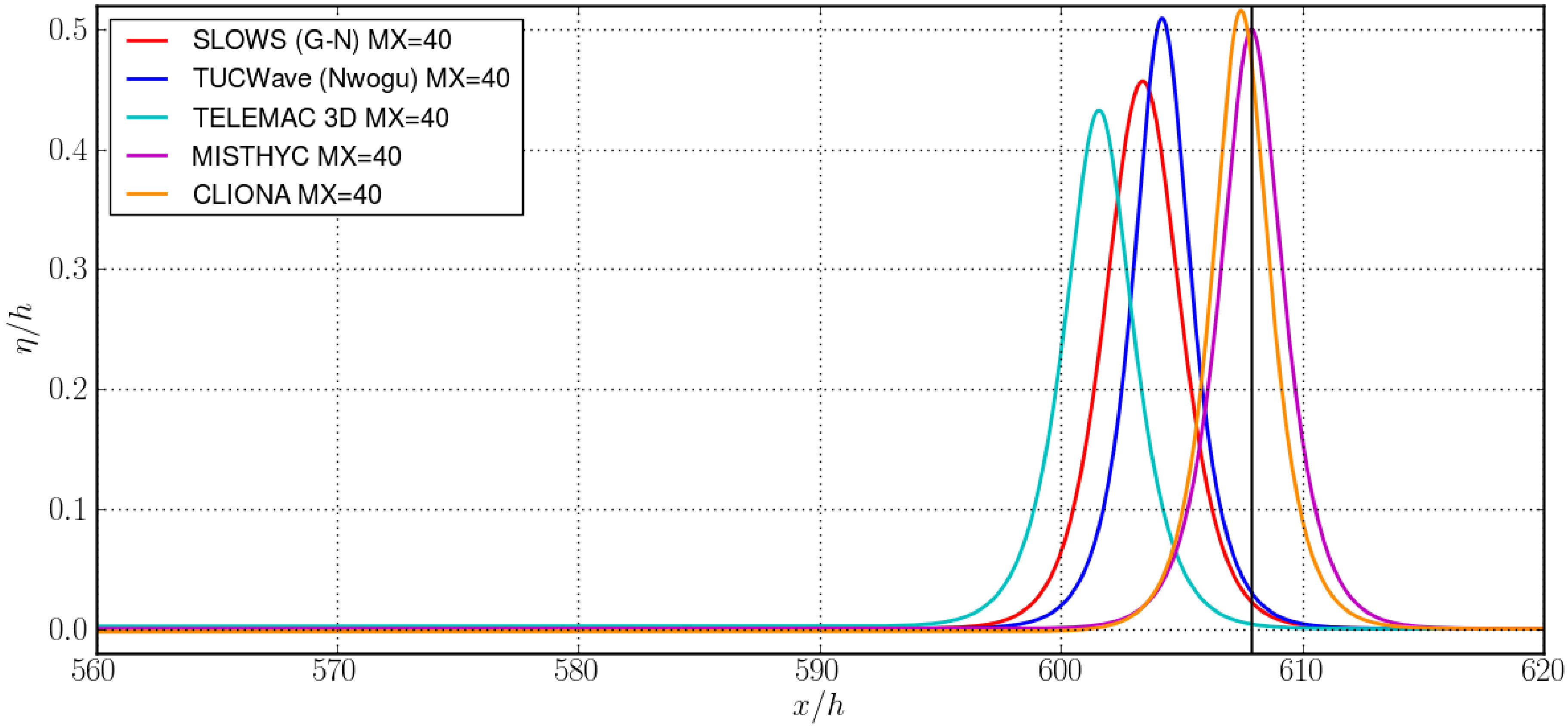
H/h = 0.5 – Free surface at final time – $M_x = h/\Delta x$ = 10



H/h = 0.5 – Free surface at final time – $M_x = h/\Delta x$ = 20

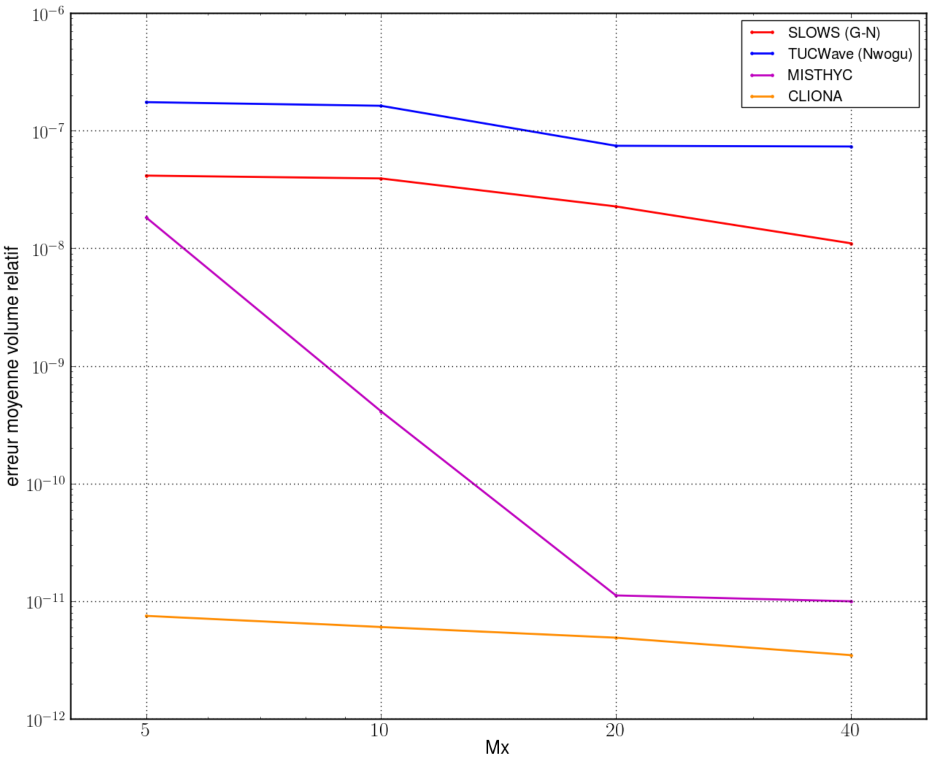


H/h = 0.5 – Free surface at final time – $M_x = h/\Delta x = 40$

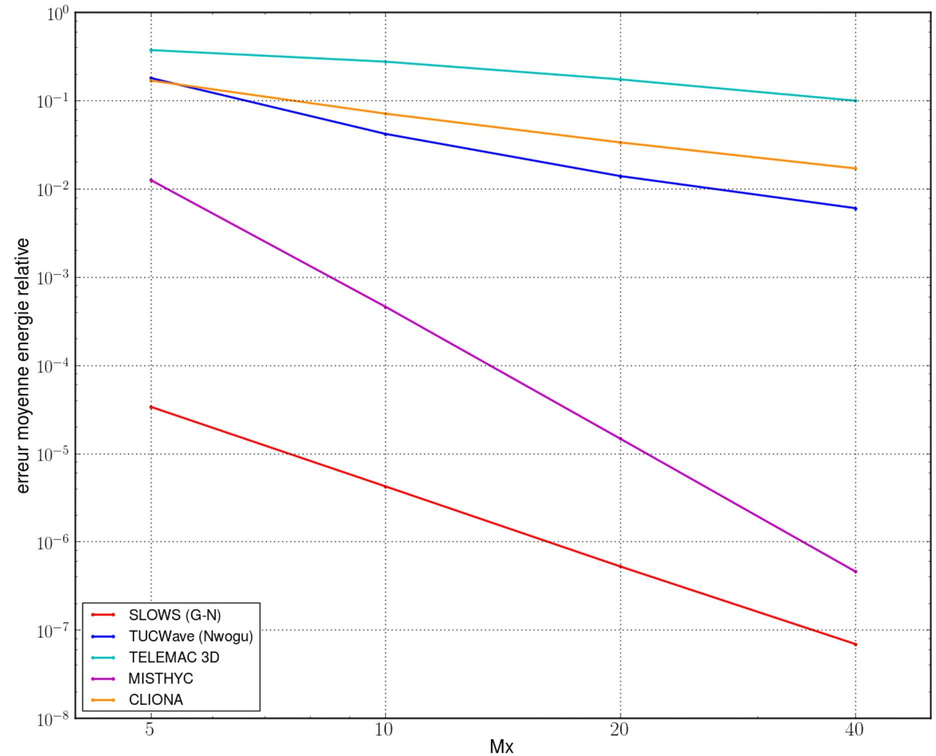


H/h = 0.5 – Mean errors (over the duration of the simulation) on conservation of **volume** and **energy**

volume



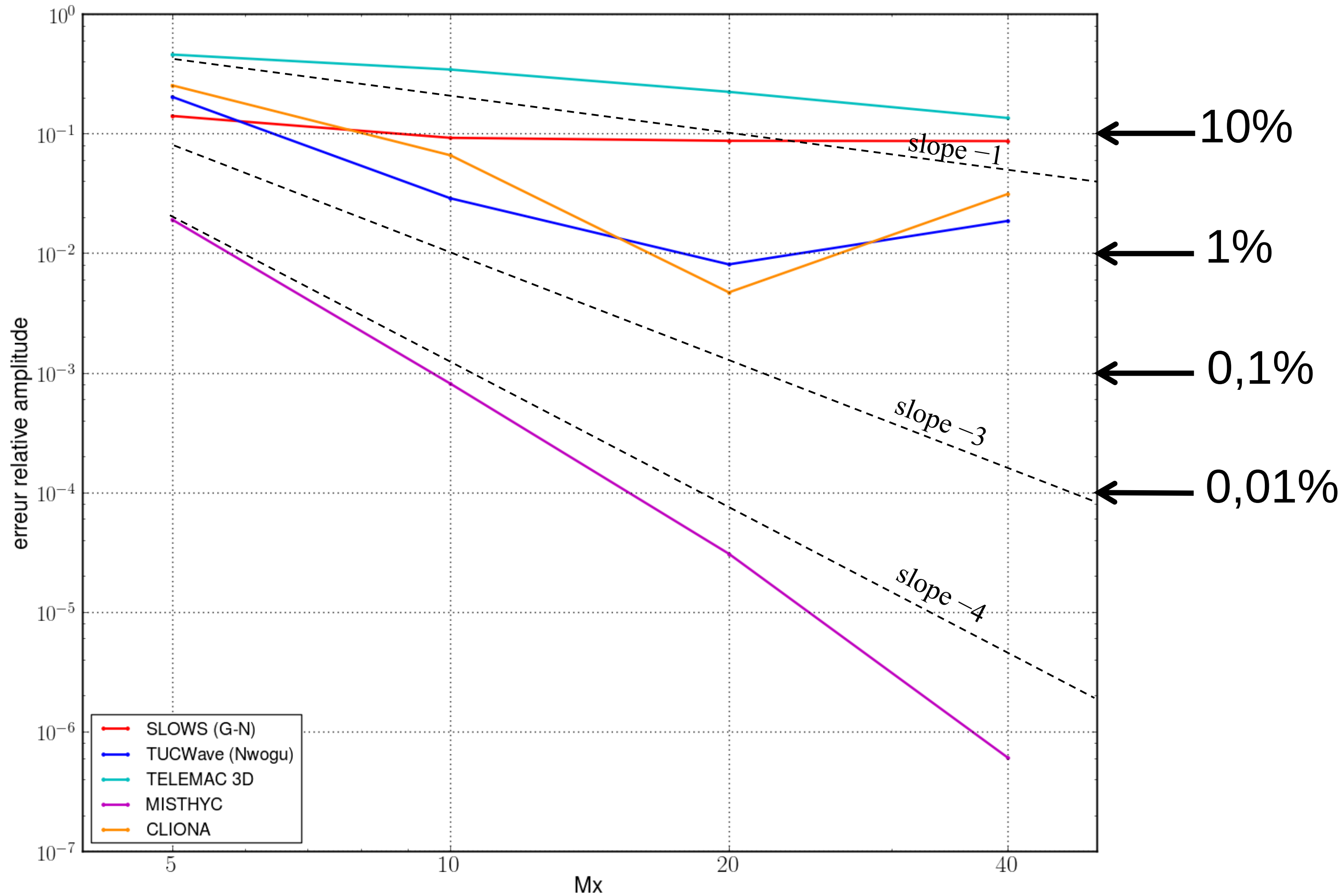
energy



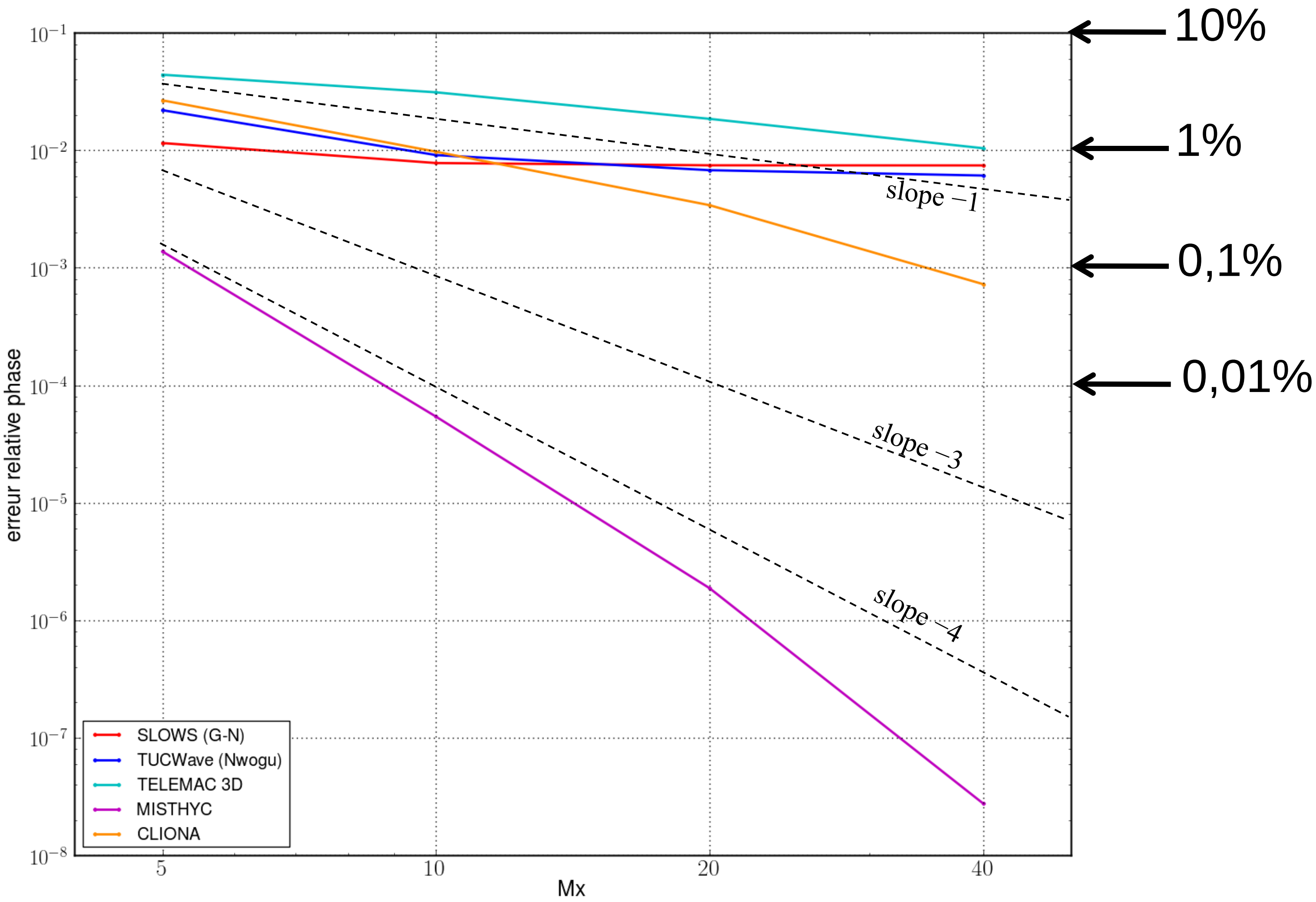
$$Err_Y = \left| \left\langle \frac{Y(t) - Y_0}{Y_0} \right\rangle_t \right| = \left| \frac{\langle Y(t) \rangle_t - Y_0}{Y_0} \right|,$$

where $\langle Y(t) \rangle_t = \frac{1}{NDT} \sum_{i=1}^{NDT} Y(t_i)$, NDT is the number of time steps, and $Y = E$ or V.

H/h = 0.5 – Error on wave **height** at final time $Err_{ampl} = \left| \frac{\eta_{max}(T) - H}{H} \right|$



H/h = 0.5 – Error on wave **position** at final time $Err_{phase} = \left| \frac{x_{max}(T) - CT}{CT} \right|$



Provisional conclusions, based on current comparisons for the case H/h = 0.5

S**ignificant differences in accuracy and quality** of results from the various codes:

1. **Misthyc** - Lowest errors on position and amplitude, whatever the spatial resolution in x Errors decay faster than $(\Delta x)^4$, e.g error in position decays from 0.14% (M_x = 5) to 2.6 10^{-8} (M_x = 40). Uses the largest Δt (CFL = 1.25) with the RK4 scheme.

2. **CLIONA (replacing earlier results from CALYPSO-Bous) –** Significantly improved results compared with former CALYPSO-Bous. Few % of errors on the wave height and phase.

3. **SLOWS (G-N)** – Results seem relatively insensitive to the horizontal resolution, indicating that the code has good convergence properties with respect to the horizontal resolution. The height of the solitary wave from SLOWS at final time is about 0.457 (relative error of about 9%). Errors on the position in the order of 1%: the wave travels slower than the theoretical wave. Excellent conservation of energy.

4. **TUCwave (Nwogu)** – Amplitude is either underestimated (M_x = 5 & 10) or overestimated (M_x = 20 & 40), with a minimum error of about 1% (M_x = 20). In all four cases, the wave travels slower than the theoretical solution.

5. **Telemac-3D –** Results have improved compared to Dec. 2014, with new tuning of options and numerical parameters. But still large errors both in amplitude with attenuation between 46% (M_x = 5) and 13% (M_x = 40) and position (significant delay compared to the theoretical solution, error always > 1% of the total travelled distance).

A **journal paper is being prepared** on this test-case, including H/h = 0.3 and 0.7 cases.

Part V – A new test-case: long-distance propagation, shoaling and run-up of a rectangular wave

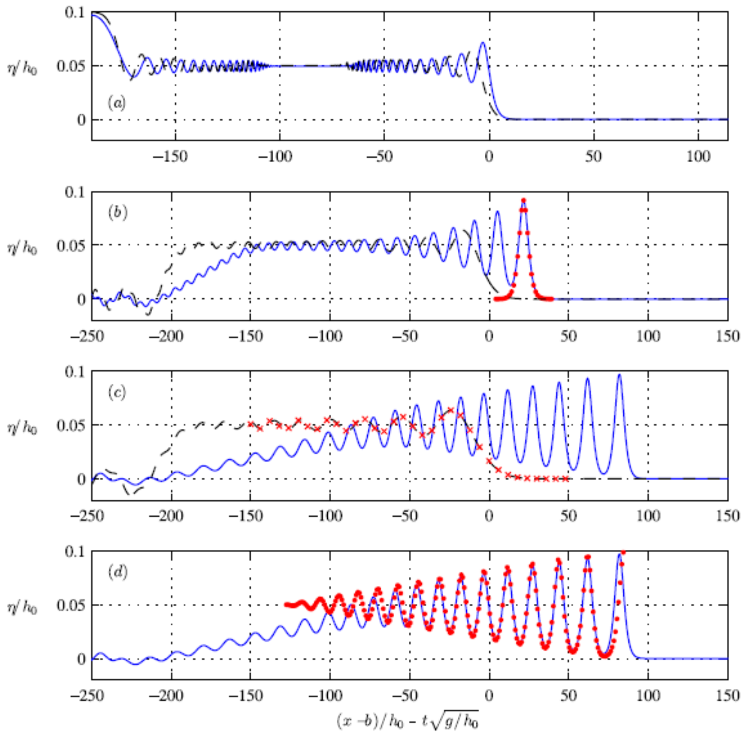


Figure 2. Computed surface elevations evolving from a positive initial rectangular hump of water (height: a, width: $2b$). Scales: $a/h_0 = 0.10$, $b/h_0 = 100$. (a) $t\sqrt{g/h_0} = 90$; (b) $t\sqrt{g/h_0} = 700$; (c) $t\sqrt{g/h_0} = 2000$; (d) $t\sqrt{g/h_0} = 2000$. Full line: nonlinear dispersive model; dashed line: linear dispersive model; cross: theoretical linear infinite width solution according to equation (22); dots on b: solitary wave solution; dots on d: the theoretical solution by *Gurevich and Pitaevskii* [1974] determined at time $t\sqrt{g/h_0} = 1700$.

From Madsen et al. (2008)

TANDEM project - Work-package WP1

P04 - Long-distance propagation, shoaling and run-up of a rectangular wave

Goal: compare the results of many solvers, in particular regarding the run-up/run-down of the tsunami wave train at boundaries of the domain (min/max elevations and associated times).

- Combine 3 phases of a tsunami:
  (i) propagation, (ii) shoaling and (iii) run-up on vertical wall.
- 2DV case, that can be modelled with 3D codes assuming invariance along the transverse direction (use 1 or 2 cells in the transverse direction).
- Choice of realistic characteristics (water depth, distances, initial deformation)
- Simple boundary and initial conditions (instantaneous deformation of the free surface with analytical shape, without any velocity) => easy to set up and run.
- Reference results could be obtained from highly accurate Euler solver from F. Dias (Matlab code used in Viotti and Dias (2014))
- Could be run by all codes, from NLSWE codes to Navier-Stokes solvers
  => everybody should (in principle) be able to run it.

P04 - Test-case definition

Assumptions and general settings:

- Inviscid fluid of constant and homogeneous density (incompressible flow).
- Constant and homogeneous atmospheric pressure at the free surface.
- Surface tension effects neglected.
- Acceleration of gravity $g = 9.81 \text{ m/s}^2$

Bathymetry:

- Fully defined analytically on a 30 km long domain (see figures) and constant in time
- Two profiles considered, including or not a 10 km long deep water pit.
- Impermeable bottom, with a slip condition $\underline{u}.\underline{n} = 0$

Lateral boundary conditions:

- Vertical fully reflective (impermeable) walls, with a slip condition $\underline{u}.\underline{n} = 0$

Initial conditions:

- Initial free-surface elevation with a (smoothed) rectangular shape, fully defined analytically:

  $0 \leq x < 1\,900$ m $\eta(x) = 5$ m

  $1\,900 \leq x \leq 2\,100$ m $\eta(x) = 2.5\,(1 - \tanh(\Delta(x/2000 - 1)))$ with $\Delta = 229.756$

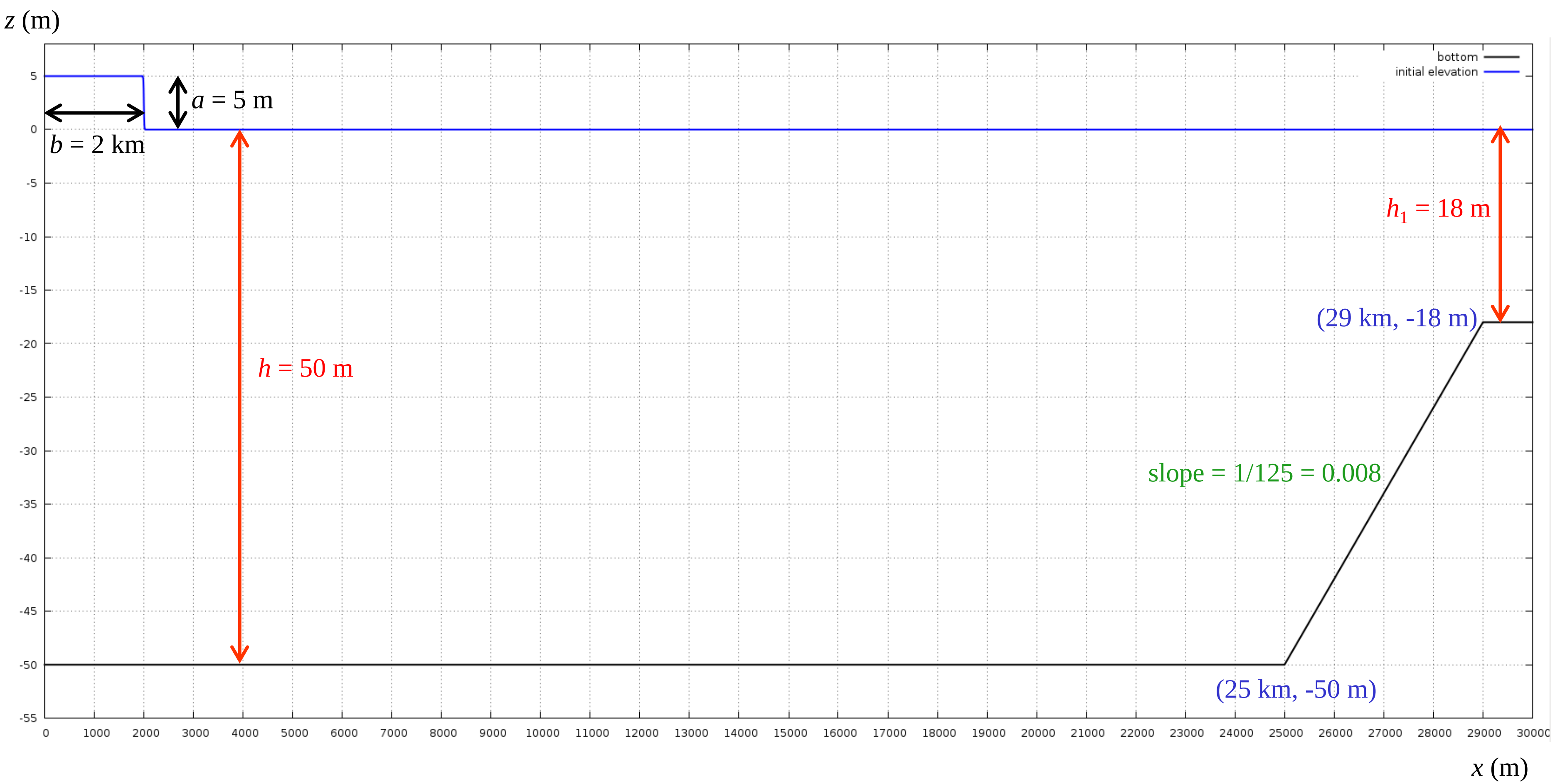
  $2\,100 < x \leq 30\,000$ m $\eta(x) = 0$

- Fluid at rest in the whole domain at $t = 0$ ($\underline{u} = \underline{0}$ everywhere).

Simulated duration: 1 h = 3600 s (with Δt left to participant's choice).

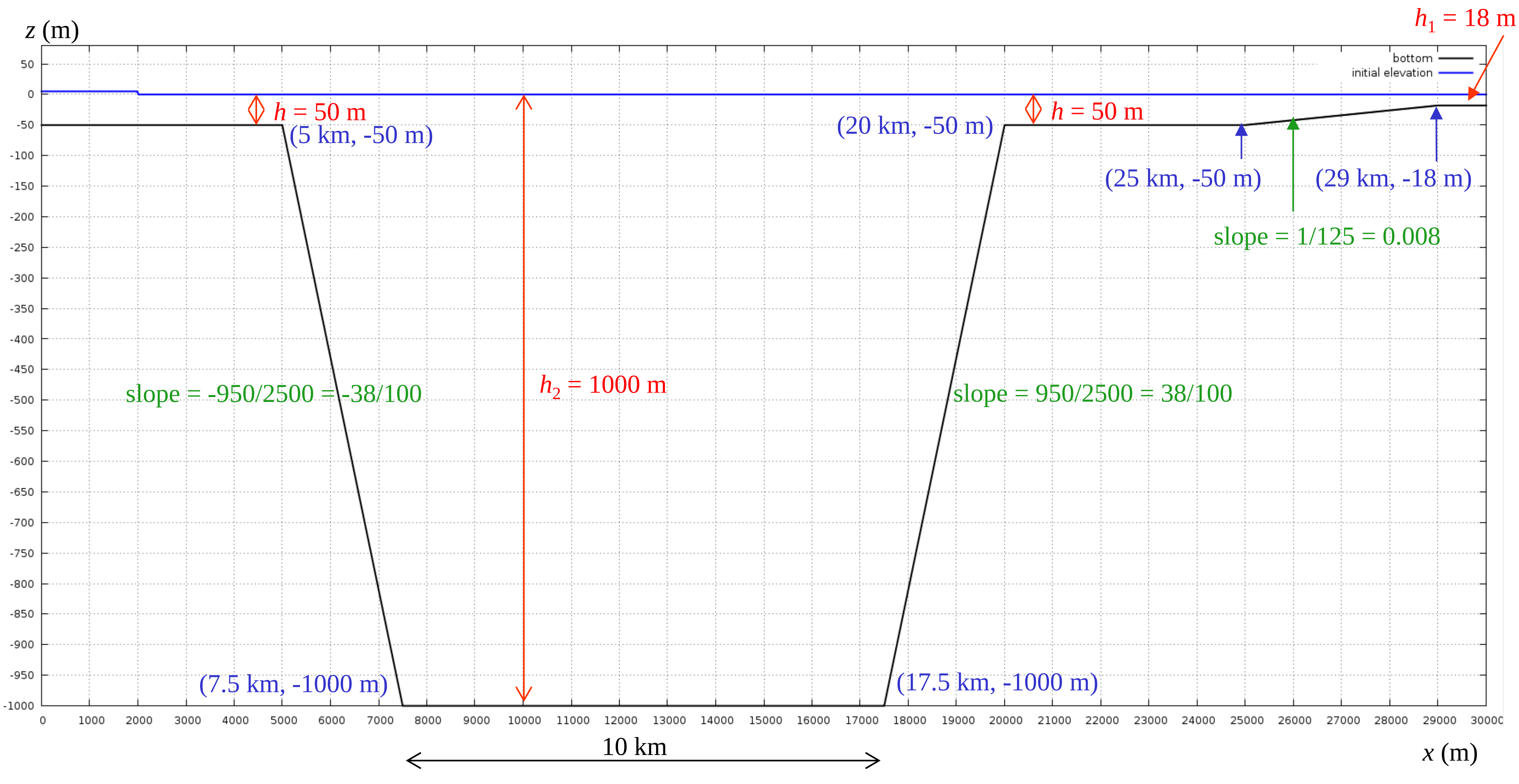
P04 - Test-case definition – Bottom profile 1

Profile 1 : constant depth (h = 50 m) + slope on the right + horizontal shelf with h_1 = 18 m



P04 - Test-case definition – Bottom profile 2

Deep water pit (h_2 = 1000m) in the domain (5 km < x < 20 km), otherwise identical to profile 1.



Participants to this (ongoing) benchmark

Ongoing test-case => PROVISIONAL RESULTS
Results up to 21 min (before run-up of the right wall): propagation + shoaling

Nonlinear shallow water codes (Saint-Venant equations):
(in principle non-dispersive and nonlinear models)

- Telemac-2D (EDF)
- Saint-Venant solver of EoleNS (Principia)

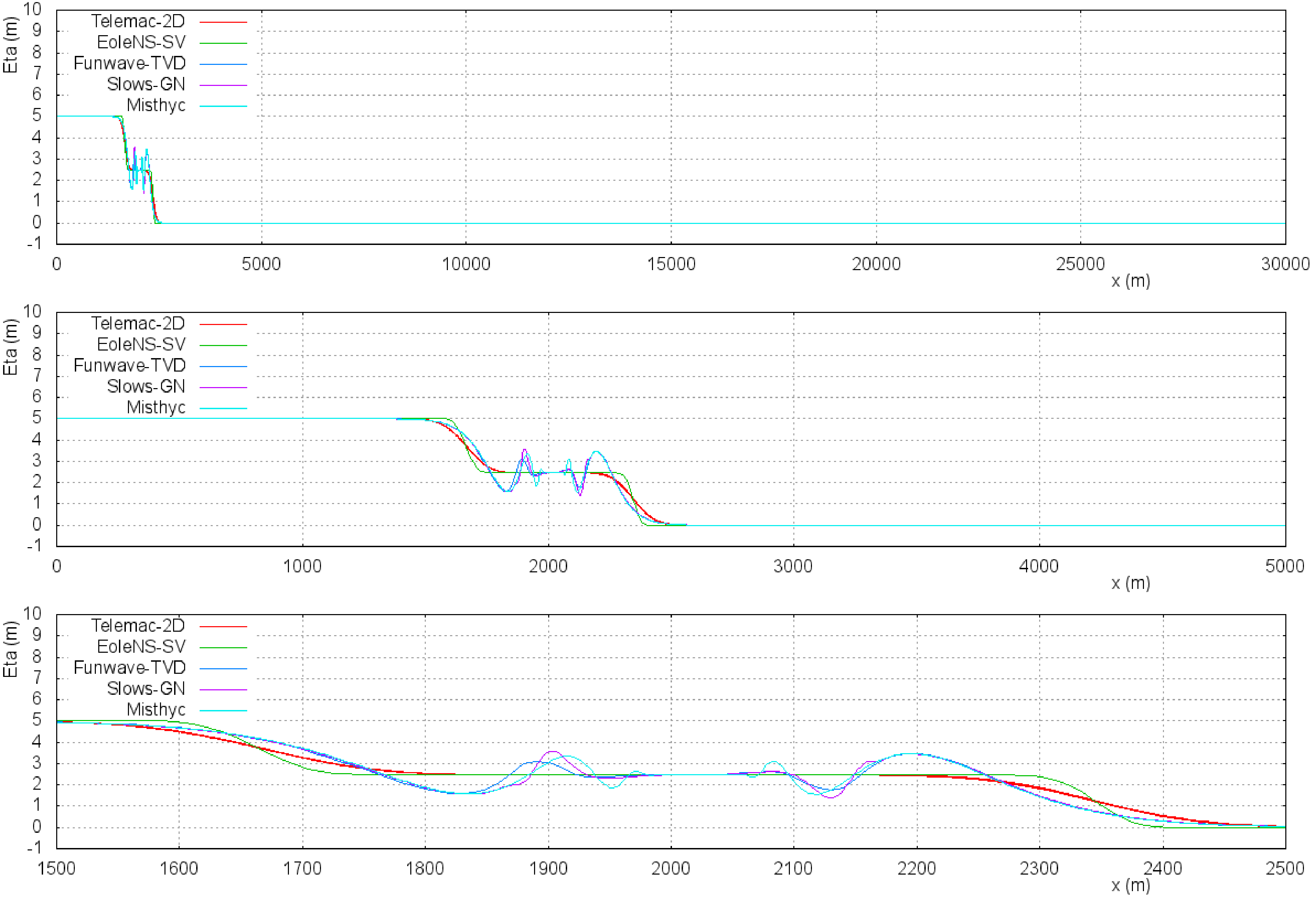
Boussinesq and Green-Naghdi codes:
(in principle weakly dispersive and weakly nonlinear models)

- Funwave-TVD (BRGM)
- SLOWS-GN (INRIA Bordeaux)
- *+ possibly (not received yet): CALYPSO and/or CLIONA (CEA), Telemac-2D Boussinesq (EDF), TUCwaves (INRIA Bordeaux)*

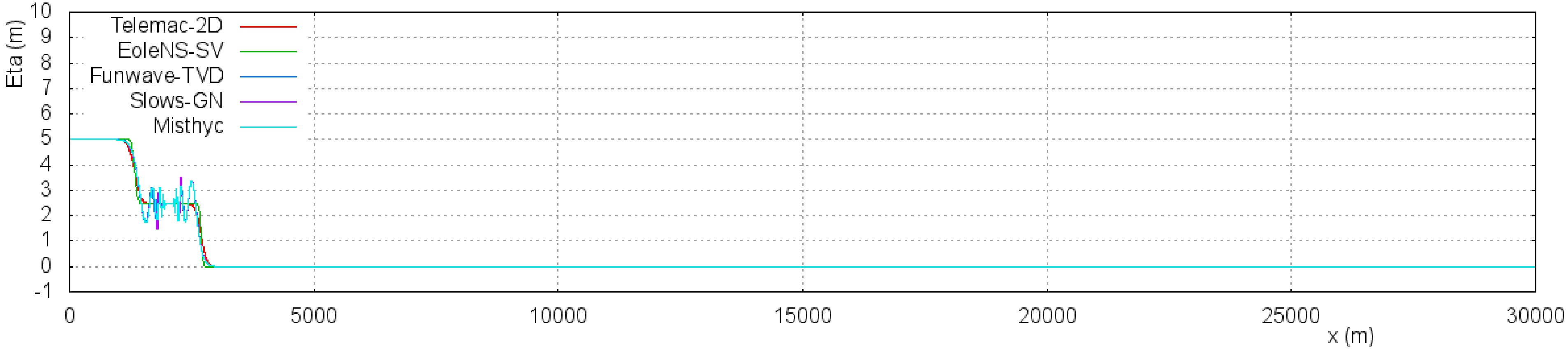
Euler potential and Navier-Stokes (without viscosity) codes:
(in principle fully dispersive and fully nonlinear models)

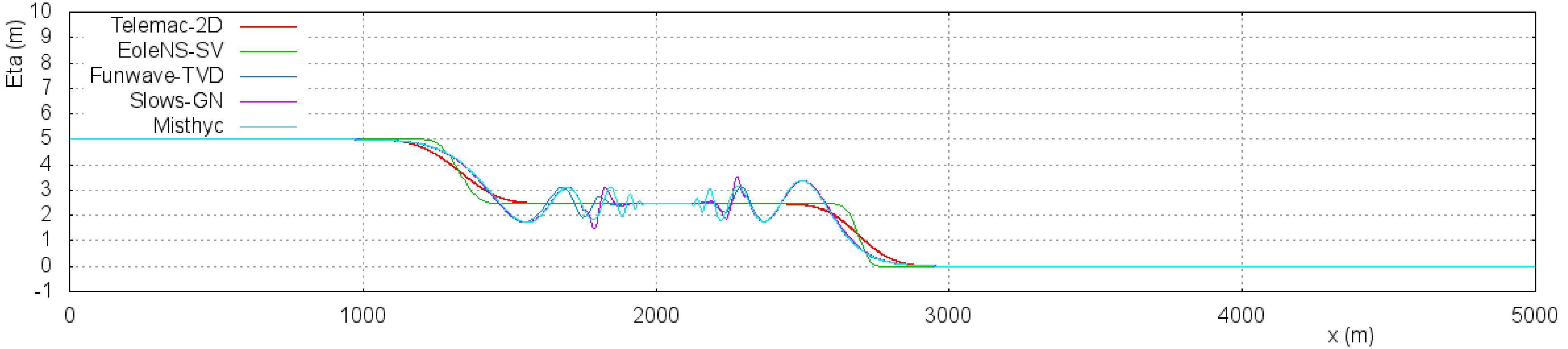
- Misthyc (IRPHE & Saint-Venant Lab.)
- *+ possibly (not received yet): Telemac-3D (EDF)*

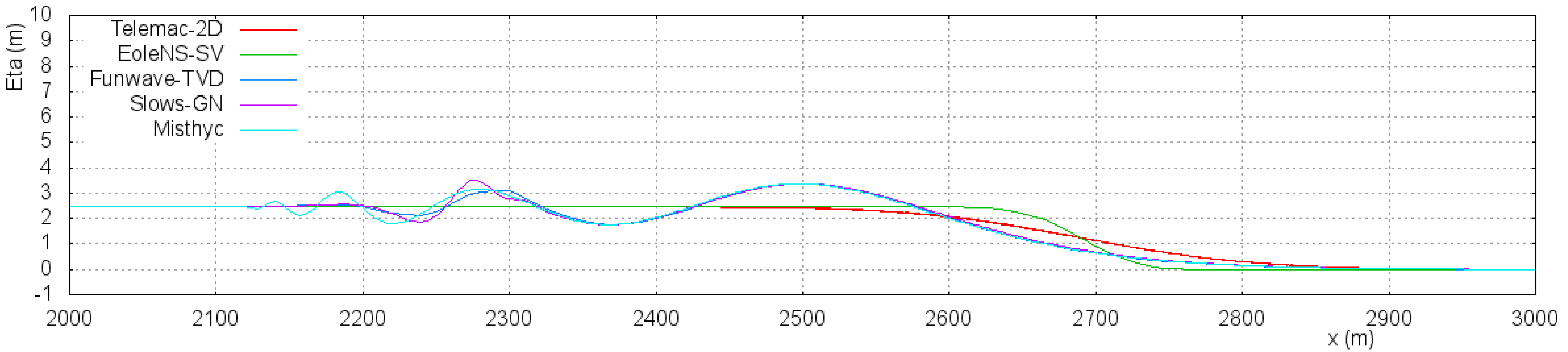
P04 – profile 1 – Results at t = 15 s – PROVISIONAL RESULTS



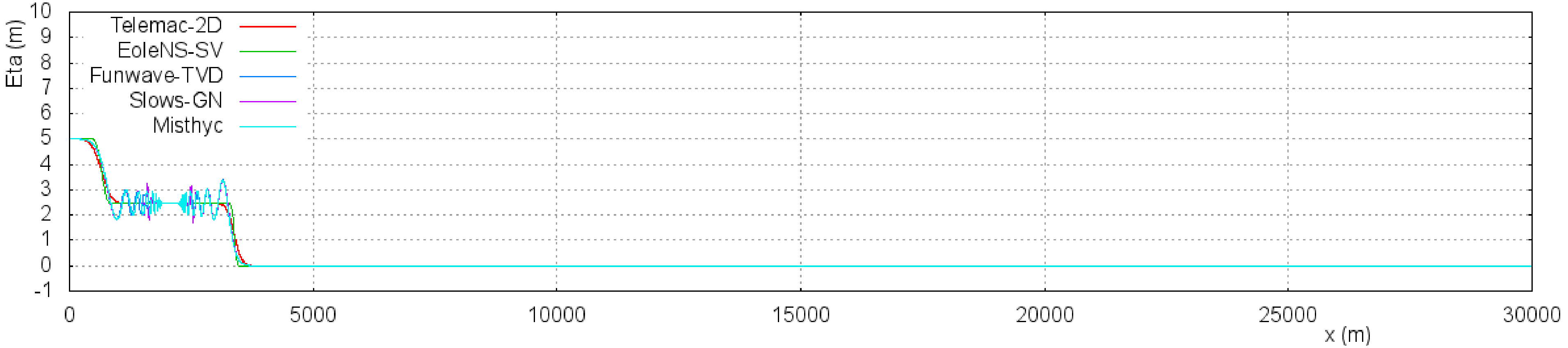
P04 – profile 1 – Results at t = 30 s – PROVISIONAL RESULTS

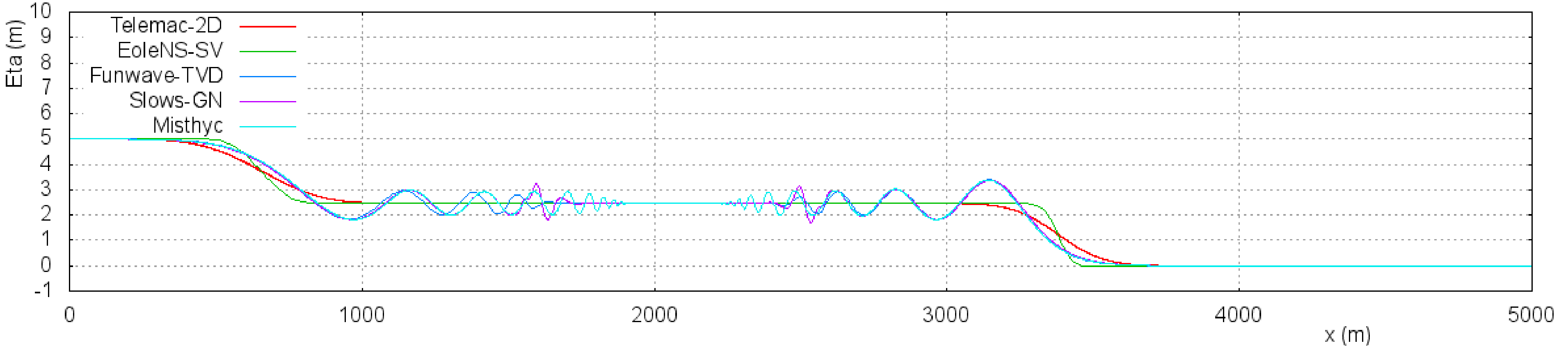


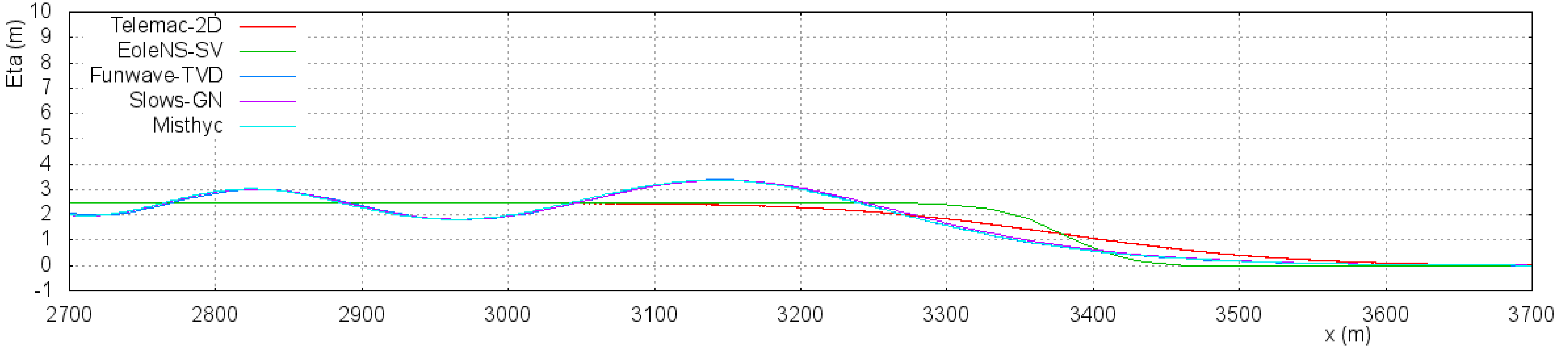




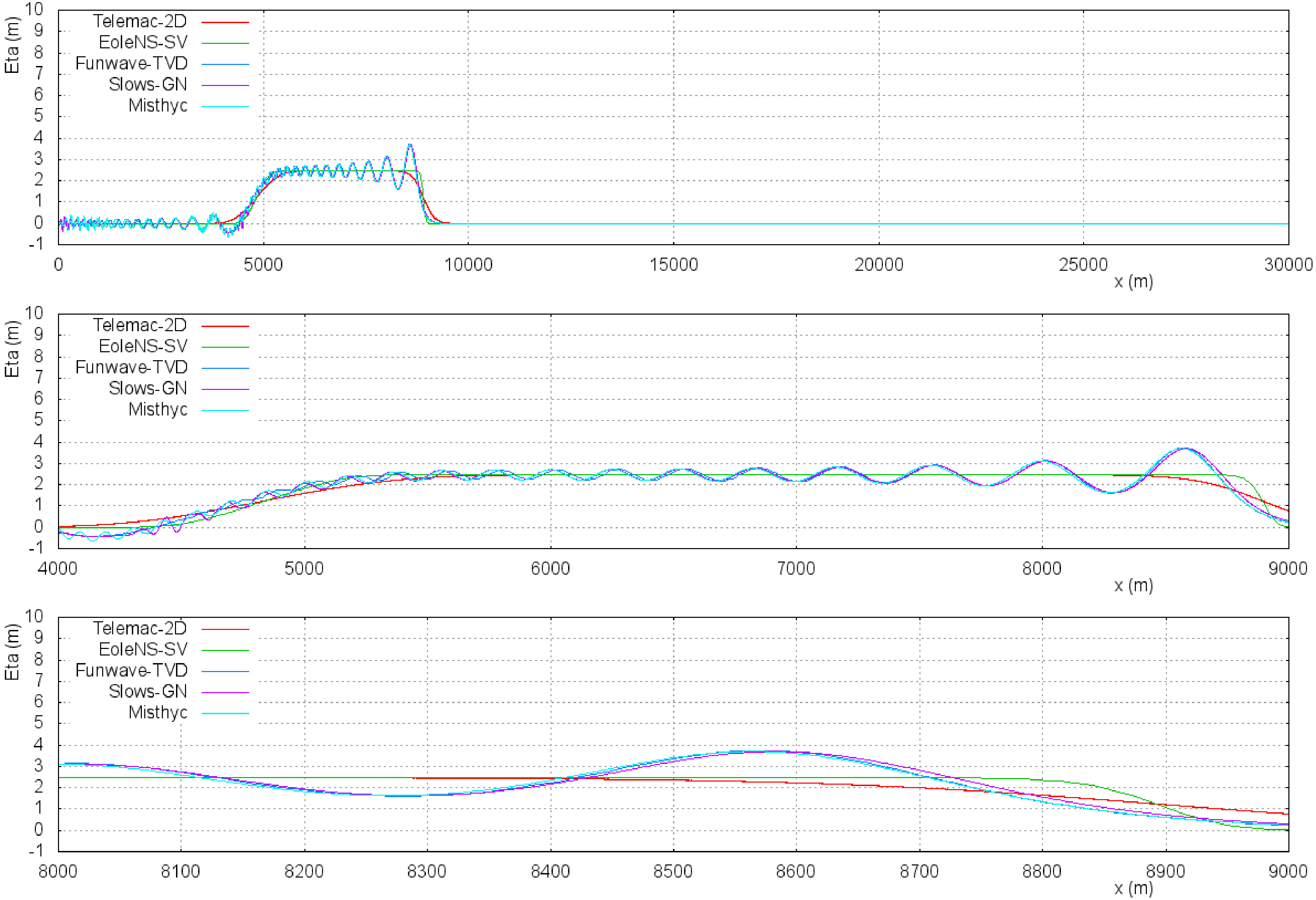
P04 – profile 1 – Results at t = 1 min – PROVISIONAL RESULTS



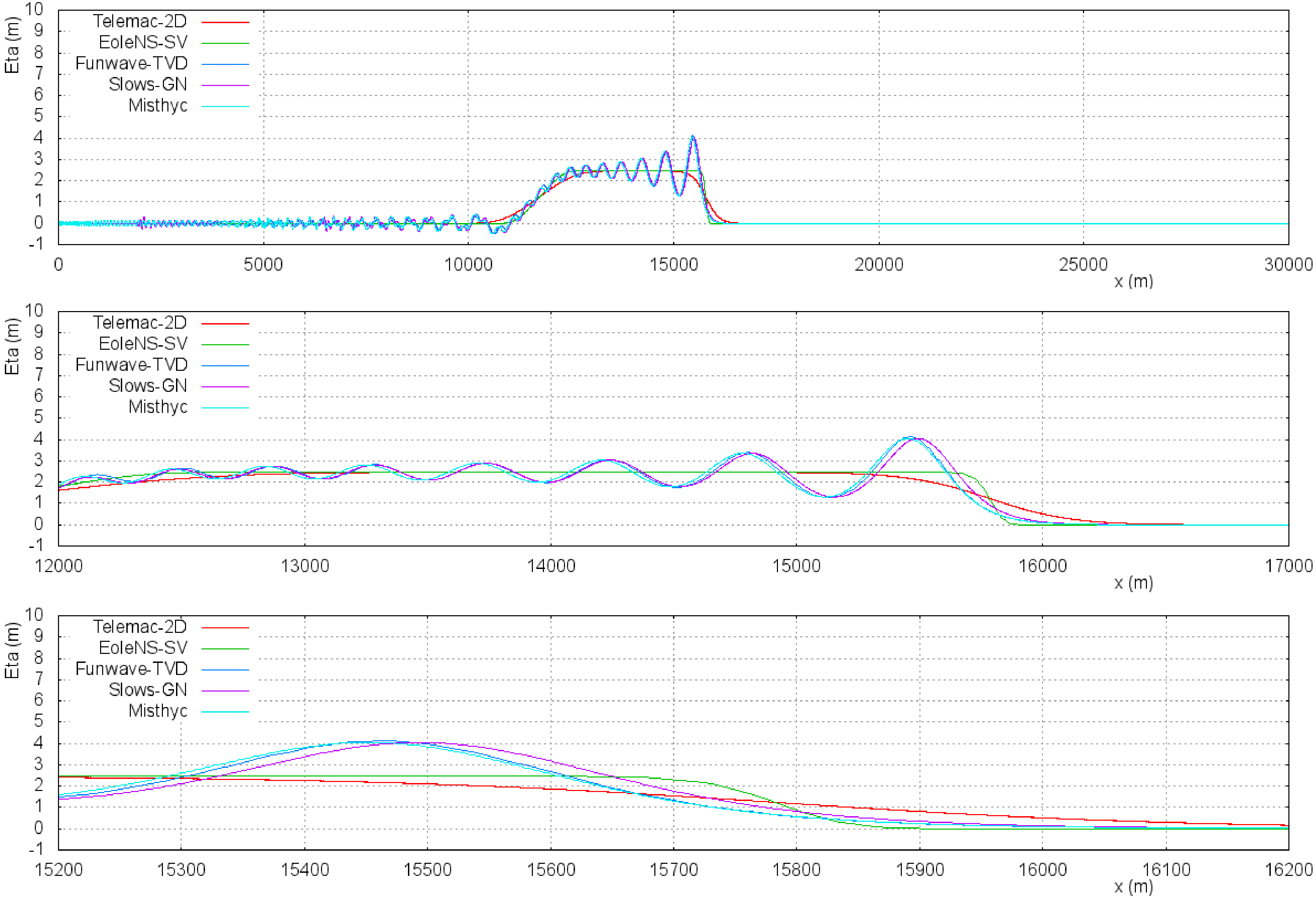




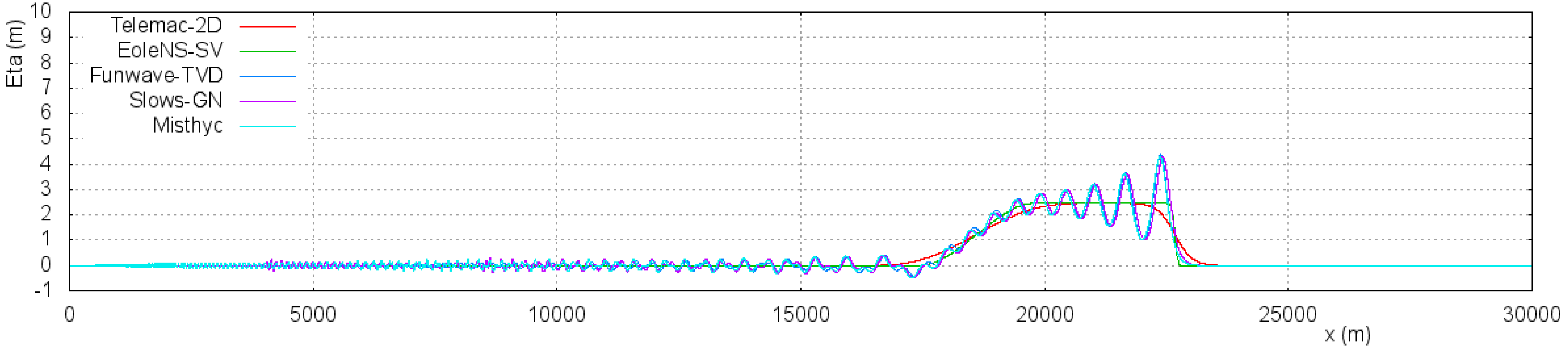
P04 – profile 1 – Results at t = 5 min – PROVISIONAL RESULTS

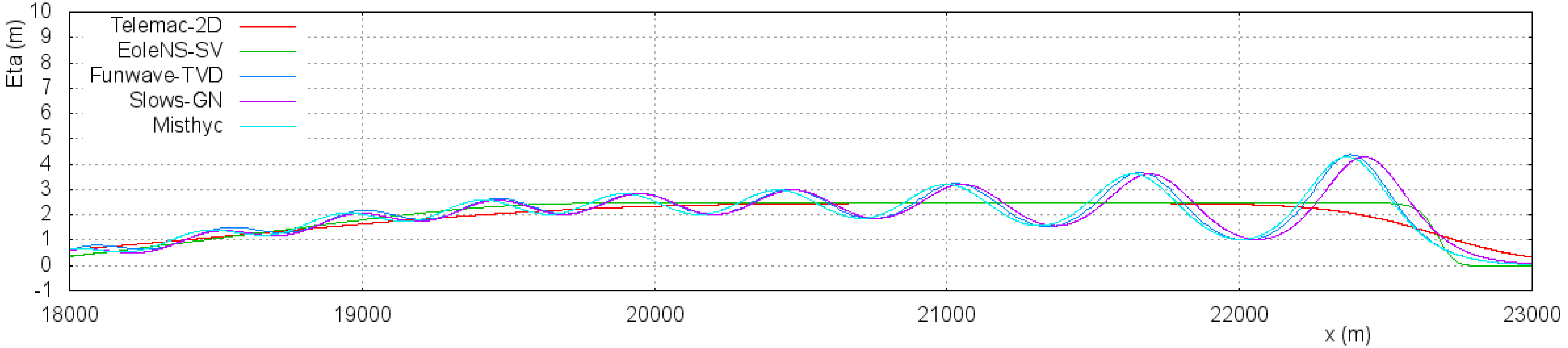


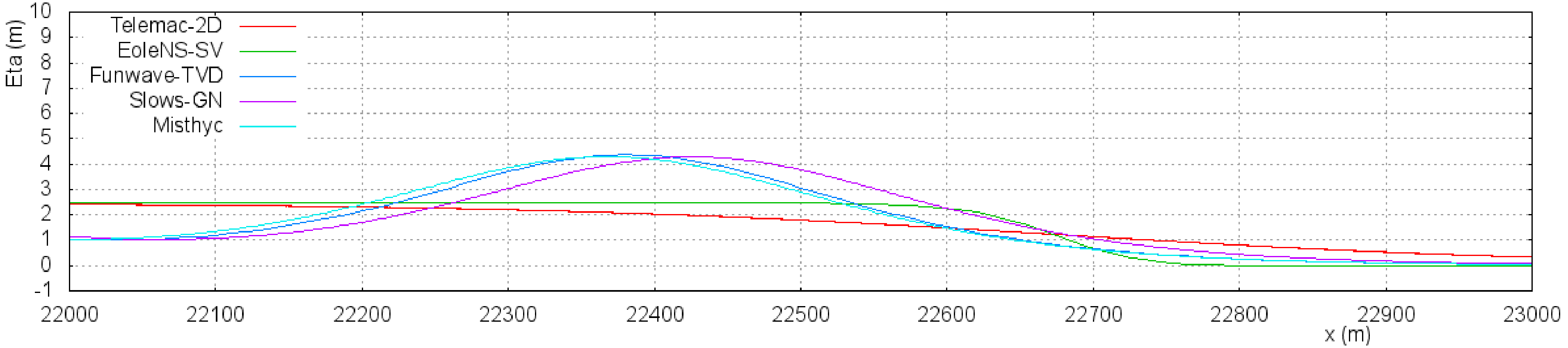
P04 – profile 1 – Results at t = 10 min – PROVISIONAL RESULTS



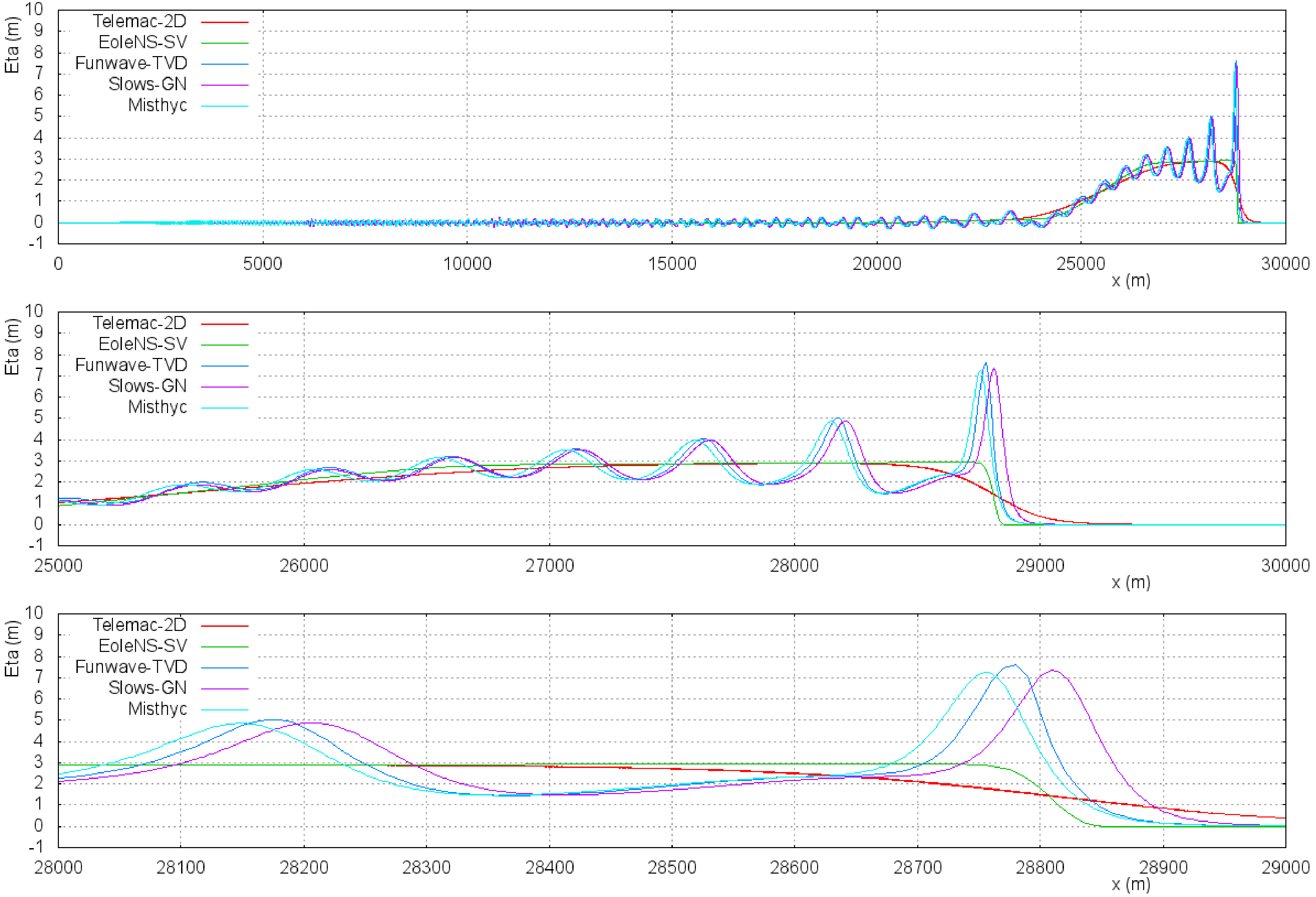
P04 – profile 1 – Results at t = 15 min – PROVISIONAL RESULTS



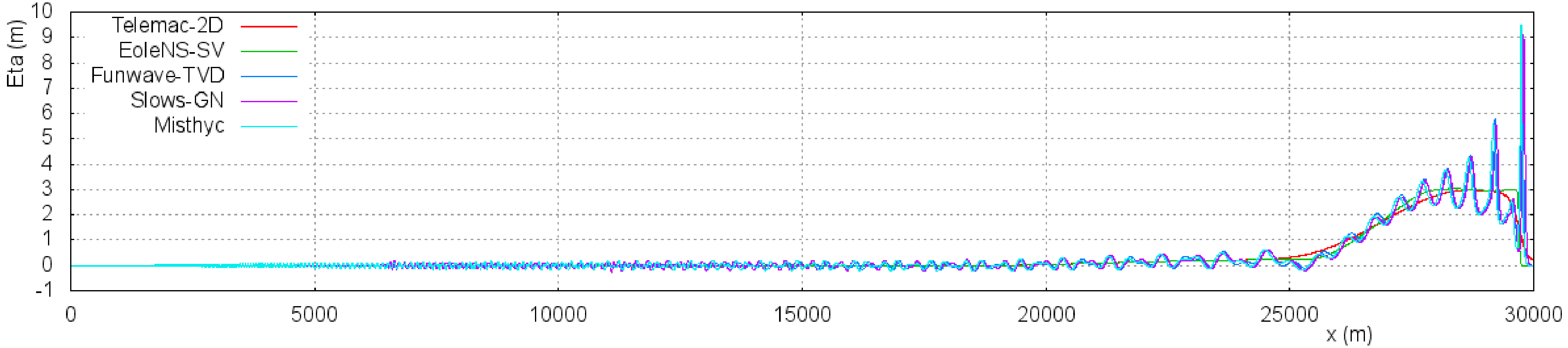


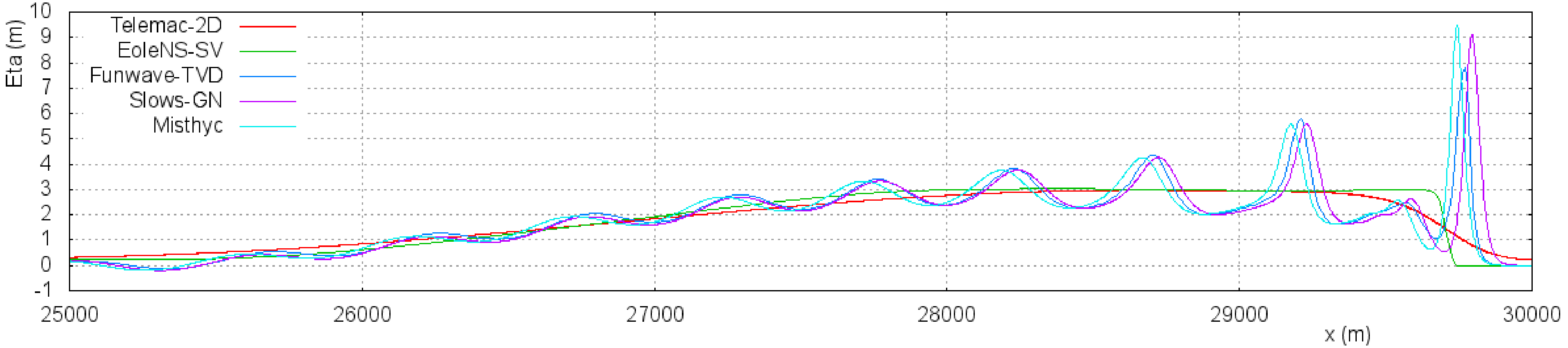


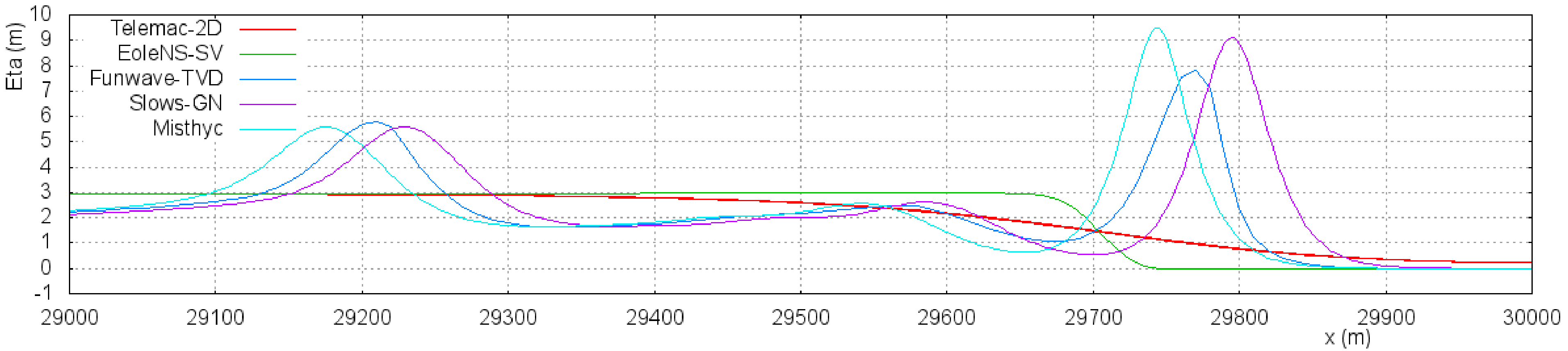
P04 – profile 1 – Results at t = 20 min – PROVISIONAL RESULTS



P04 – profile 1 – Results at t = 21 min – PROVISIONAL RESULTS







As a matter of conclusions (1)

From a physical perspective:

- Dispersive effects can be significant:
  - during the generation and propagation of tsunami, in particular for landslide tsunamis and small to medium seismic tsunamis.
    The analysis of the dispersion time τ has shown the source width (or initial wavelength for landslides) λ is more important for the significance of dispersion than the depth h or propagation distance L.

    $$\tau = \frac{6h^2 L}{\lambda^3}$$
  - in the shoaling area when the tsunami wave approaches the shore, giving rise to undular bore type of waves.
- Nonlinear effects:
  - Usually negligible for large scale tsunamis in depth water (ε = H/h small) during the generation and propagation stages.
    (=> linear shallow water eq. could be used in this case).
  - Should however be considered for landslide tsunamis or seismic tsunamis in shallow water (moderate to low values of depth h).
  - becomes significant in the shoaling area (nonlinear amplification) in decreasing h.
- Some mechanisms, such as soliton fission (during propagation) and undular bore shape (close to the coast), can only be captured if both dispersive and nonlinear are taken into account.

As a matter of conclusions (2)

From a numerical modelling perspective:

- Never forget that what you get as output of a simulation code is the result of
  (math./physical model) x (numerical schemes/options) x (discretization choices)
  Differences with measurements/reference solutions may come from any of these components (all of them simultaneously most of time…)

- Important to perform **detailed verification and validation studies** to appreciate errors associated with math. and num. model, and sensitivity/convergence studies to assess errors due to discretization of the model.
  => *purpose of the WP1 of TANDEM project*.

- **Depending on the type and phase of the tsunami, dispersive and nonlinear effects are more or less significant.** If one wishes to solve the full dynamics of a tsunami from the generation to the coast, **it is recommended to use models that are at least mildly dispersive and nonlinear, e.g. Boussinesq-type or Serre-Green-Naghdi or, better, higher-order models.**

- Of course, **computational efficiency may be important** (e.g. warning systems), and this can limit the level of nonlinearity/dispersion of the model
  => **compromise to find between accuracy and efficiency**.

These remain largely open questions… Work to be continued !

Thank you !