Numerical issues 1 : well-balancing, positivity, high order.

Arnaud Duran
INSA Toulouse

Joint work with F. Marche

Tsunami school, Bordeaux
27 / 04 / 2016

2D Formulation

$$\frac{\partial}{\partial t}U + \nabla . G(U) = B(U,z)\,.$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \; G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2}gh^2 + hu^2 & huv \\ huv & \frac{1}{2}gh^2 + hv^2 \end{pmatrix}, \; B(U,z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Coastal hydrodynamic

2D Formulation

$$\frac{\partial}{\partial t} U + \nabla . G(U) = B(U, z) \, .$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \; G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2}gh^2 + hu^2 & huv \\ huv & \frac{1}{2}gh^2 + hv^2 \end{pmatrix}, \; B(U, z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Coastal hydrodynamic

2D Formulation

$$\frac{\partial}{\partial t}U + \nabla.G(U) = B(U,z)\,.$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix},\ G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2}gh^2 + hu^2 & huv \\ huv & \frac{1}{2}gh^2 + hv^2 \end{pmatrix},\ B(U,z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Coastal hydrodynamic

2D Formulation

$$\frac{\partial}{\partial t}U + \nabla . G(U) = B(U, z) \, .$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} , \; G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2}gh^2 + hu^2 & huv \\ huv & \frac{1}{2}gh^2 + hv^2 \end{pmatrix} , \; B(U, z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Coastal hydrodynamic

2D Formulation

$$\frac{\partial}{\partial t} U + \nabla . G(U) = B(U, z) \, .$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} , \; G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2} g h^2 + hu^2 & huv \\ huv & \frac{1}{2} g h^2 + hv^2 \end{pmatrix} , \; B(U, z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Tsunamis

2D Formulation

$$\frac{\partial}{\partial t}U + \nabla . G(U) = B(U,z)\,.$$

$$U = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \; G(U) = \begin{pmatrix} hu & hv \\ \frac{1}{2}gh^2 + hu^2 & huv \\ huv & \frac{1}{2}gh^2 + hv^2 \end{pmatrix}, \; B(U,z) = \begin{pmatrix} 0 \\ -gh\partial_x z \\ -gh\partial_y z \end{pmatrix}$$

Application field : some examples

Rivers,
dam breaks

Stability criteria

- Preservation of steady states :
  → **(C-property)** [Bermudez & Vázquez, 1994]

$$h + z = cte\,, \mathbf{u} = 0\,.$$

- Robustness : preservation of the water depth positivity.
- Entropy inequalities.

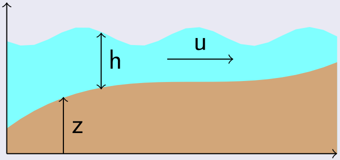
Notable advances :

▷ [Greenberg, Leroux, 1996] , [Gosse, Leroux , 1996] scalar case
▷ [Garcia-Navarro, Vázquez-Cendón , 1997] , [Castro, Gonzales, Pares, 2006] Roe schemes, [LeVeque, 1998] *wave-propagation* algorithm
▷ [Perthame, Simeoni, 2001] , [Perthame, Simeoni, 2003] *kinetic schemes*
▷ [LeVeque, 2002] , [Bouchut, 2004] *Stability of FVM for Hyperbolic Conservation Laws*
▷ [Gallouët, Hérard, Seguin, 2003] , [Berthon, Marche, 2008] *VFRoe schemes*
▷ [Audusse *et al*, 2004] *Hydrostatic Reconstruction*, [Ricchiuto *et al*, 2007] RD schemes , [Lukácová-Medvidová, Noelle, Kraft, 2007] FVEG schemes

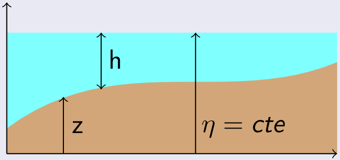
Outline

Outline

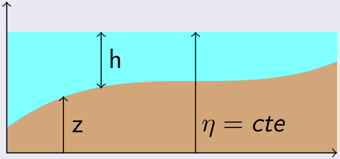
1D Configuration



1D Lake at rest configuration



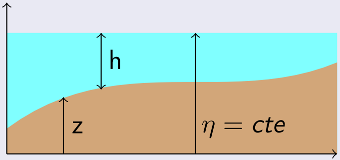
1D Lake at rest configuration



First works :

- [Zhou, Causon, Mingham, 2001] *Surface Gradient Method*
- [Rogers, Fujihara, Borthwick, 2001] , [Russo, 2005] , [Xing, Shu, 2005]

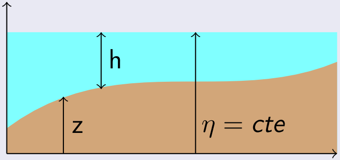
1D Lake at rest configuration



First works :

- ▷ [Zhou, Causon, Mingham, 2001] *Surface Gradient Method*
- ▷ [Rogers, Fujihara, Borthwick, 2001] , [Russo, 2005] , [Xing, Shu, 2005]
- ▷ [Liang, Borthwick, 2009] , [Liang, Marche, 2009] "*Pre-Balanced*" formulation.

1D Lake at rest configuration



Pre balanced formulation

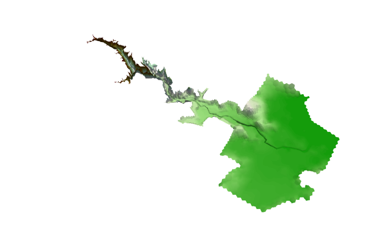
$$\frac{\partial}{\partial t}V + \nabla . H(V,z) = S(V,z)\,.$$

$$V = \begin{pmatrix} \eta \\ hu \\ hv \end{pmatrix} \;,\; H(V,z) = \begin{pmatrix} hu & hv \\ \frac{1}{2}g(\eta^2 - 2\eta z) + hu^2 & huv \\ huv & \frac{1}{2}g(\eta^2 - 2\eta z) + hv^2 \end{pmatrix}.$$

Topography source term : $\quad S(V,z) = \begin{pmatrix} 0 \\ -g\eta\partial_x z \\ -g\eta\partial_y z \end{pmatrix}.$

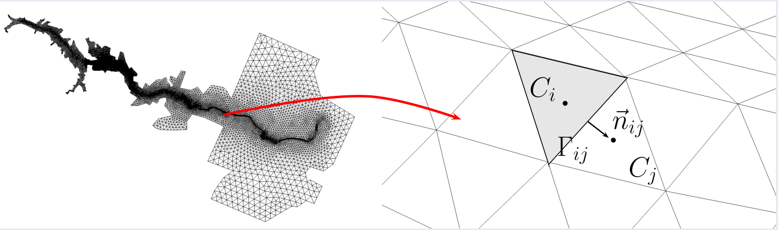
Numerical issues 1
Arnaud Duran
Shallow Water Equations
Finite Volume approach : Stability issues
Reformulation of SW equations
Finite Volumes : generalities
Accounting for bottom variations
Characteristics
MUSCL extension
Some results
dG extension
Extension to dispersive equations
Perspectives

Geometry and unstructured meshes



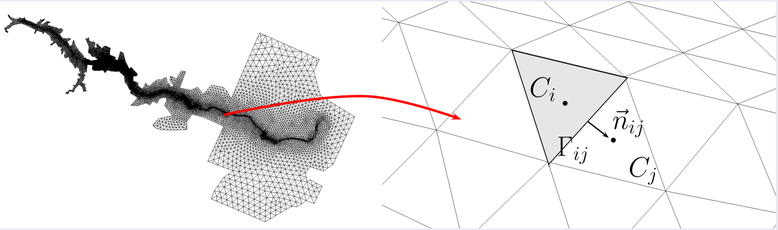
Finite Volume formulation

Geometry and unstructured meshes



Finite Volume formulation

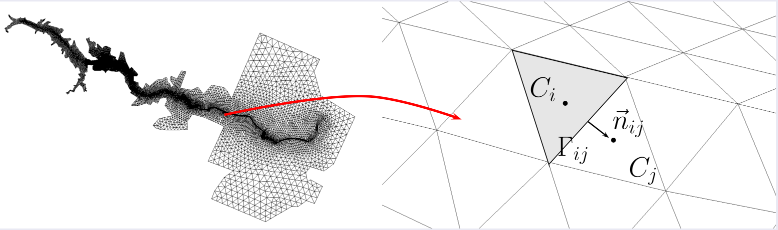
Geometry and unstructured meshes



Finite Volume formulation

$$\frac{\partial}{\partial t}V + \nabla.H(V,z) = S(V,z)\,.$$

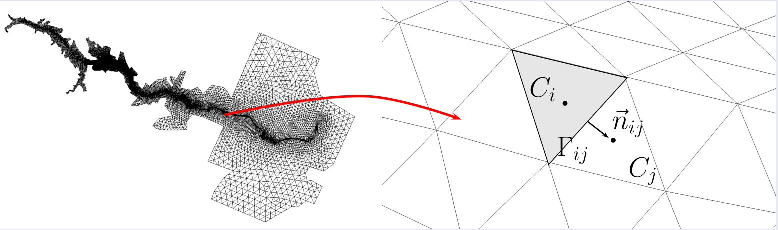
Geometry and unstructured meshes



Finite Volume formulation

$$\int_{C_i} \frac{\partial}{\partial t} V \, d\mathbf{x} + \int_{C_i} \nabla . H(V,z) \, ds = \int_{C_i} S(V,z) \, d\mathbf{x} \, .$$

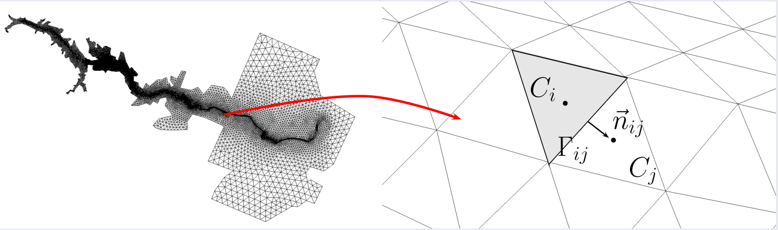
Geometry and unstructured meshes



Finite Volume formulation

$$\int_{C_i} \frac{\partial}{\partial t} V \, \mathbf{dx} + \int_{\partial C_i} H(V, z).\vec{n} \, ds = \int_{C_i} S(V, z) \mathbf{dx} \, .$$

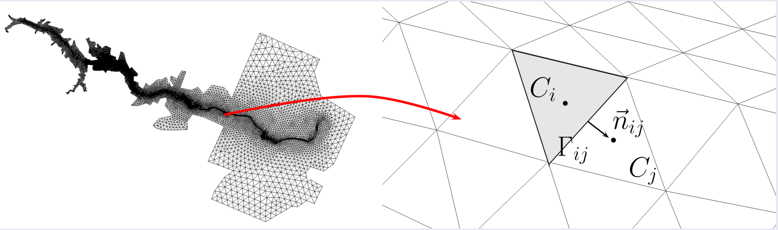
Geometry and unstructured meshes



Finite Volume formulation

$$\int_{C_i} \frac{\partial}{\partial t} V \, d\mathbf{x} + \sum_{k=1}^{\Lambda(i)} \int_{\Gamma_{ij(k)}} H(V, z).\vec{n}_{ij(k)} \, ds = \int_{C_i} S(V, z) d\mathbf{x} \, .$$

Geometry and unstructured meshes

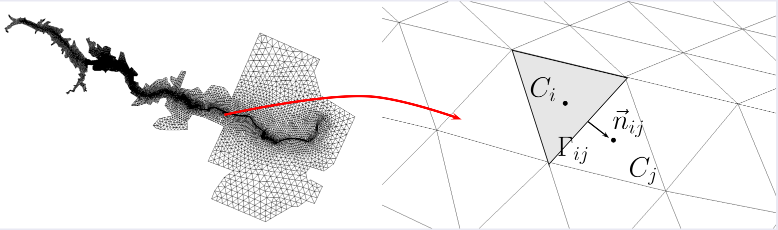


Finite Volume formulation

$$|C_i|\frac{d}{dt}V_i + \sum_{k=1}^{\Lambda(i)} \int_{\Gamma_{ij(k)}} H(V,z).\vec{n}_{ij(k)}\,ds = \int_{C_i} S(V,z)d\mathbf{x}\,,$$

$$V_i = \frac{1}{|C_i|}\int_{C_i} V(\mathbf{x},t)d\mathbf{x}\,.$$

Geometry and unstructured meshes

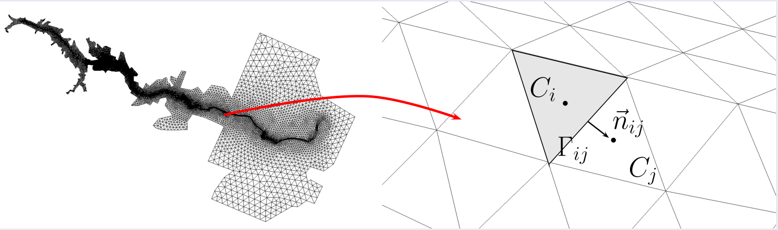


Finite Volume formulation

$$|C_i|\frac{d}{dt}V_i + \sum_{k=1}^{\Lambda(i)} \int_{\Gamma_{ij(k)}} H(V,z).\vec{n}_{ij(k)}\, ds = \int_{C_i} S(V,z)d\mathbf{x}\,.$$

Numerical fluxes : $\displaystyle\int_{\Gamma_{ij(k)}} H(V,z).\vec{n}_{ij(k)}\, ds \approx \ell_{ij(k)}\mathcal{H}_{ij(k)}\,.$

Geometry and unstructured meshes



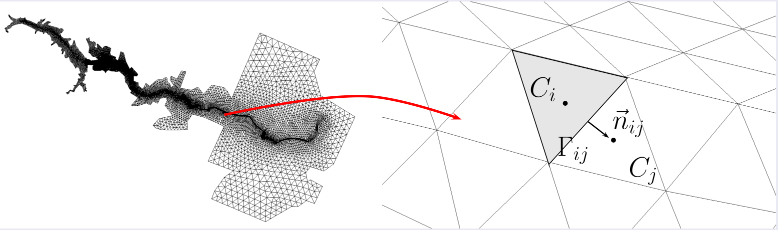
Finite Volume formulation

$$|C_i|\frac{d}{dt}V_i + \sum_{k=1}^{\Lambda(i)} \int_{\Gamma_{ij(k)}} H(V,z).\vec{n}_{ij(k)}\, ds = \int_{C_i} S(V,z)d\mathbf{x}\,.$$

$$|C_i|\frac{d}{dt}V_i + \sum_{k=1}^{\Lambda(i)} \underbrace{\ell_{ij(k)}\mathcal{H}_{ij(k)}}_{\text{Numerical fluxes}} = S_i\,.$$

Generalities

Geometry and unstructured meshes

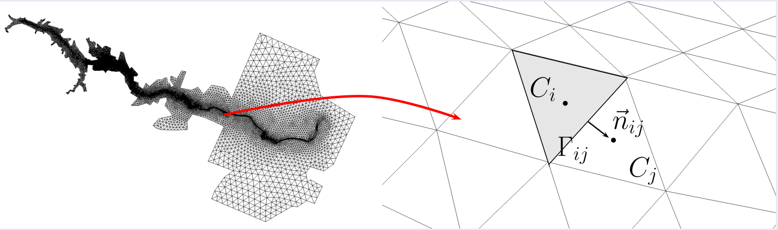


Finite Volume formulation

$$\cancel{|C_i| \frac{d}{dt} V_i} + \sum_{k=1}^{\Lambda(i)} \int_{\Gamma_{ij(k)}} H(V,z).\vec{n}_{ij(k)}\, ds = \int_{C_i} S(V,z) d\mathbf{x}\,.$$

$$|C_i| \frac{d}{dt} V_i + \sum_{k=1}^{\Lambda(i)} \underbrace{\ell_{ij(k)} \mathcal{H}_{ij(k)}}_{\text{Numerical fluxes}} = S_i\,.$$

Geometry and unstructured meshes



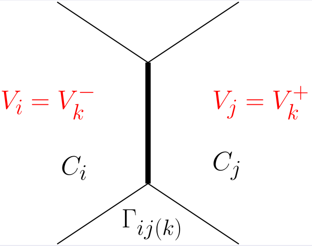
Finite Volume formulation

$$\cancel{|C_i|\frac{d}{dt}V_i} + \sum_{k=1}^{\Lambda(i)} \int_{\Gamma_{ij(k)}} H(V,z).\vec{n}_{ij(k)}\, ds = \int_{C_i} S(V,z)d\mathbf{x}\,.$$

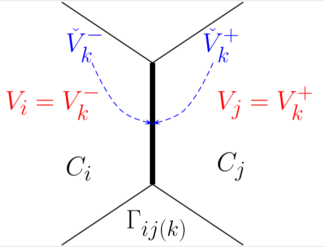
$$\cancel{|C_i|\frac{d}{dt}V_i} + \sum_{k=1}^{\Lambda(i)} \underbrace{\ell_{ij(k)}\mathcal{H}_{ij(k)}}_{\text{Numerical fluxes}} = S_i\,.$$

Evaluation of the numerical fluxes

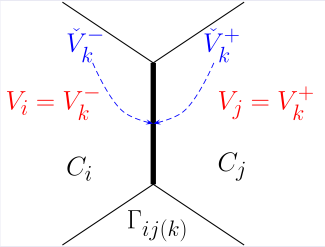
Reconstruction at the interfaces



Reconstruction at the interfaces

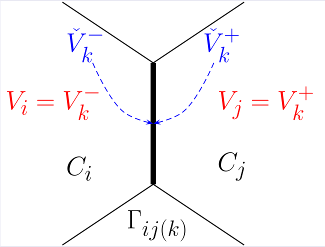


Reconstruction at the interfaces



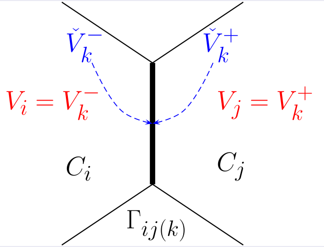
$$\tilde{z}_k = \max(z_k^-, z_k^+)\,,$$

Reconstruction at the interfaces



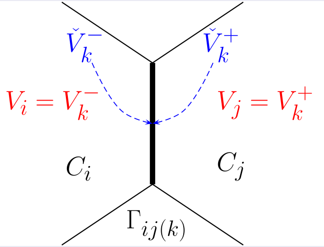
$$\tilde{z}_k = \max(z_k^-, z_k^+)\,,$$
$$\check{h}_k^- = \max(0, \eta_k^- - \tilde{z}_k)\,,$$
$$\check{h}_k^+ = \max(0, \eta_k^+ - \tilde{z}_k)\,,$$

Reconstruction at the interfaces



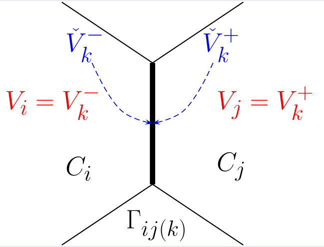
$$\tilde{z}_k = \max(z_k^-, z_k^+)\,,$$
$$\check{h}_k^- = \max(0, \eta_k^- - \tilde{z}_k)\,,$$
$$\check{h}_k^+ = \max(0, \eta_k^+ - \tilde{z}_k)\,,$$
$$\check{z}_k = \tilde{z}_k - \max(0, \tilde{z}_k - \eta_k^-)\,,$$

Reconstruction at the interfaces



$$\tilde{z}_k = \max(z_k^-, z_k^+)\,,$$
$$\breve{h}_k^- = \max(0, \eta_k^- - \tilde{z}_k)\,,$$
$$\breve{h}_k^+ = \max(0, \eta_k^+ - \tilde{z}_k)\,,$$
$$\breve{z}_k = \tilde{z}_k - \max(0, \tilde{z}_k - \eta_k^-)\,,$$
$$\breve{\eta}_k^- = \breve{h}_k^- + \breve{z}_k \quad , \quad \breve{\mathbf{q}}_k^- = \frac{\breve{h}_k^-}{h_k^-}\mathbf{q}_k^-\,,$$
$$\breve{\eta}_k^+ = \breve{h}_k^+ + \breve{z}_k \quad , \quad \breve{\mathbf{q}}_k^+ = \frac{\breve{h}_k^+}{h_k^+}\mathbf{q}_k^+\,.$$

Reconstruction at the interfaces

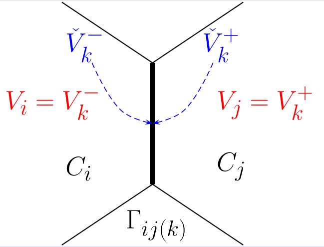


$$\tilde{z}_k = \max(z_k^-, z_k^+)\,,$$
$$\check{h}_k^- = \max(0, \eta_k^- - \tilde{z}_k)\,,$$
$$\check{h}_k^+ = \max(0, \eta_k^+ - \tilde{z}_k)\,,$$
$$\check{z}_k = \tilde{z}_k - \max(0, \tilde{z}_k - \eta_k^-)\,,$$
$$\check{\eta}_k^- = \check{h}_k^- + \check{z}_k \quad , \quad \check{\mathbf{q}}_k^- = \frac{\check{h}_k^-}{h_k^-}\mathbf{q}_k^-\,,$$
$$\check{\eta}_k^+ = \check{h}_k^+ + \check{z}_k \quad , \quad \check{\mathbf{q}}_k^+ = \frac{\check{h}_k^+}{h_k^+}\mathbf{q}_k^+\,.$$

[E. Audusse *et al*, 2004] *Hydrostatic reconstruction*

Evaluation of the numerical fluxes

Reconstruction at the interfaces



$$\tilde{z}_k = \max(z_k^-, z_k^+)\,,$$
$$\check{h}_k^- = \max(0, \eta_k^- - \tilde{z}_k)\,,$$
$$\check{h}_k^+ = \max(0, \eta_k^+ - \tilde{z}_k)\,,$$
$$\check{z}_k = \tilde{z}_k - \max(0, \tilde{z}_k - \eta_k^-)\,,$$
$$\check{\eta}_k^- = \check{h}_k^- + \check{z}_k \quad , \quad \check{\mathbf{q}}_k^- = \frac{\check{h}_k^-}{h_k^-}\mathbf{q}_k^-\,,$$
$$\check{\eta}_k^+ = \check{h}_k^+ + \check{z}_k \quad , \quad \check{\mathbf{q}}_k^+ = \frac{\check{h}_k^+}{h_k^+}\mathbf{q}_k^+\,.$$

[E. Audusse *et al*, 2004] *Hydrostatic reconstruction*

General formulation

$$\mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$

▷ **Numerical scheme :**

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{|C_i|} \sum_{k=1}^{\Lambda(i)} \ell_{ij(k)} \mathcal{H}_{ij(k)} + \Delta t S_i \,. \qquad (1)$$

▷ **Objective :** Preservation of the motionless steady states.
▷ **Key idea :** Involve the properties of the 1D scheme.

Approximation of the source term

$$S_i = -\frac{1}{|C_i|} \sum_{k=1}^{\Lambda(i)} \ell_{ij(k)} S_{ij(k)} = -\frac{1}{|C_i|} \sum_{k=1}^{\Lambda(i)} \ell_{ij(k)} \begin{pmatrix} 0 \\ g\hat{\eta}_k (z_i - \check{z}_k) \vec{n}_{ij(k)} \end{pmatrix} .$$

▷ Provided an appropriate choice of $\hat{\eta}_k$, the scheme (1) can be rewritten as a convex combination of 1D schemes.

The scheme written under convex combination

$$V_i^{n+1} = \sum_{k=1}^{\Lambda(i)} \frac{|T_{ij(k)}|}{|C_i|} V_{ij(k)}^{n+1},$$

$$V_{ij(k)}^{n+1} = V_i^n - \frac{\Delta t}{\Delta_{ij(k)}}\Big(\mathcal{H}_{ij(k)} - H(V_i, z_i).\vec{n}_{ij(k)}\Big) + \frac{\Delta t}{\Delta_{ij(k)}} S_{ij(k)}.$$

- Exact preservation of "lake at rest" configurations.
- Preservation of the positivity of the water depth.
- Stability and robust treatment in the neighbourhood of dry areas and wet/dry interfaces.

The scheme written under convex combination

$$V_i^{n+1} = \sum_{k=1}^{\Lambda(i)} \frac{|T_{ij(k)}|}{|C_i|} V_{ij(k)}^{n+1} \, ,$$

$$V_{ij(k)}^{n+1} = V_i^n - \frac{\Delta t}{\Delta_{ij(k)}} \Big(\mathcal{H}_{ij(k)} - H(V_i, z_i).\vec{n}_{ij(k)} \Big) + \frac{\Delta t}{\Delta_{ij(k)}} S_{ij(k)} \, .$$

- Exact preservation of "lake at rest" configurations.
- Preservation of the positivity of the water depth.
- Stability and robust treatment in the neighbourhood of dry areas and wet/dry interfaces.

The scheme written under convex combination

$$V_i^{n+1} = \sum_{k=1}^{\Lambda(i)} \frac{|T_{ij(k)}|}{|C_i|} V_{ij(k)}^{n+1},$$

$$V_{ij(k)}^{n+1} = V_i^n - \frac{\Delta t}{\Delta_{ij(k)}} \Big(\mathcal{H}_{ij(k)} - H(V_i, z_i).\vec{n}_{ij(k)} \Big) + \frac{\Delta t}{\Delta_{ij(k)}} S_{ij(k)}.$$

- Exact preservation of "lake at rest" configurations.
- Preservation of the positivity of the water depth.
- Stability and robust treatment in the neighbourhood of dry areas and wet/dry interfaces.

Numerical fluxes

$$\mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)}) \,.$$

Stability

- Well - balancing :
- Robustness.

Numerical fluxes

$$\mathcal{H}_{ij(k)} = \mathcal{H}\big(\check{V}_k^{*,-}, \check{V}_k^{*,+}, \check{z}_k^{*}, \check{z}_k^{*}, \vec{n}_{ij(k)}\big)\,.$$

[S. Camarri *et al*, 2004] *MUSCL scheme for LES on unstructured grids*

Stability

- Well - balancing :
- Robustness.

Numerical fluxes

$$\mathcal{H}_{ij(k)} = \mathcal{H}\big(\check{V}_k^{*,-}, \check{V}_k^{*,+}, \check{z}_k^{*}, \check{z}_k^{*}, \vec{n}_{ij(k)}\big)\,.$$

[S. Camarri *et al*, 2004] *MUSCL scheme for LES on unstructured grids*

Stability

- Well - balancing : ok !
- Robustness.

MUSCL extension

Numerical fluxes

$$\mathcal{H}_{ij(k)} = \mathcal{H}\big(\breve{V}_k^{*,-}, \breve{V}_k^{*,+}, \breve{z}_k^{*}, \breve{z}_k^{*}, \vec{n}_{ij(k)}\big)\,.$$

[S. Camarri *et al*, 2004] *MUSCL scheme for LES on unstructured grids*

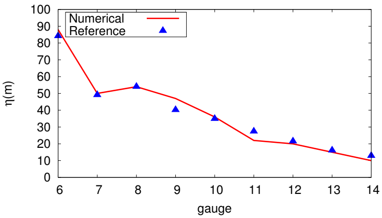
Stability

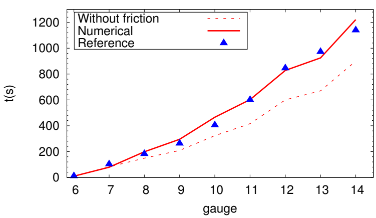
- Well - balancing : ok !
- Robustness.

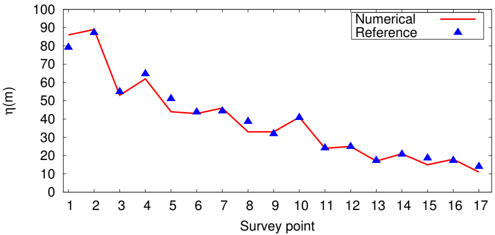
  [C. Berthon, 2006] *Robustness of MUSCL schemes for 2D unstructured meshes*

  ▷ Invoke the positivity properties of the first order scheme on appropriate subgrids.

Comparison with experimental data



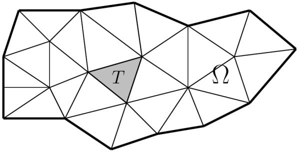




Outline

$\mathbb{P}^d(T) :=$ {2 variables polynomials on T of degree at most d} .

$\mathcal{V}_\mathfrak{h} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_\mathfrak{h}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



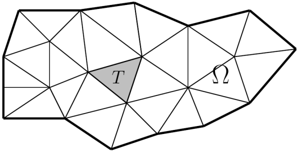
Approximate solution : $V_\mathfrak{h}(\mathbf{x}, t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Local weak formulation (1)

$$\frac{\partial}{\partial t} V + \nabla . H(V, z) = S(V, z)$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

$\mathcal{V}_\mathfrak{h} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_\mathfrak{h}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



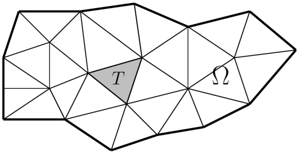
Approximate solution : $V_\mathfrak{h}(\mathbf{x}, t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Local weak formulation (2)

$$\frac{\partial}{\partial t} V\phi_\mathfrak{h}(\mathbf{x}) + \nabla.H(V,z)\phi_\mathfrak{h}(\mathbf{x}) = S(V,z)\phi_\mathfrak{h}(\mathbf{x})$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on $T$of degree at most $d\}$.

$\mathcal{V}_{\mathfrak{h}} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_{\mathfrak{h}}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



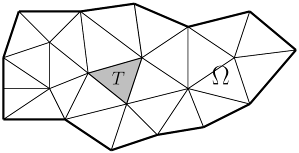
Approximate solution : $V_{\mathfrak{h}}(\mathbf{x},t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Local weak formulation (3)

$$\int_T \frac{\partial}{\partial t} V \phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} + \int_T \nabla.H(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x} = \int_T S(V,z)\phi_{\mathfrak{h}}(\mathbf{x}) d\mathbf{x}$$

$\mathbb{P}^d(T) := \{2$ variables polynomials on T of degree at most $d\}$.

$\mathcal{V}_\mathfrak{h} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_\mathfrak{h}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



Approximate solution : $V_\mathfrak{h}(\mathbf{x},t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Local weak formulation (4)

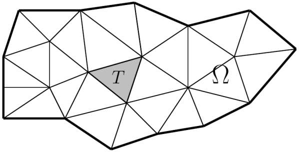
$$\int_T \frac{\partial}{\partial t} V \phi_\mathfrak{h}(\mathbf{x}) d\mathbf{x} + \boxed{\int_T \nabla . H(V,z)\phi_\mathfrak{h}(\mathbf{x}) d\mathbf{x}} = \int_T S(V,z)\phi_\mathfrak{h}(\mathbf{x}) d\mathbf{x}$$

$$\int_T \frac{\partial}{\partial t} V \phi_\mathfrak{h}(\mathbf{x}) d\mathbf{x} - \int_T H(V,z).\nabla\phi_\mathfrak{h}(\mathbf{x}) d\mathbf{x} +$$

$$\int_{\partial T} H(V,z).\vec{n}\phi_\mathfrak{h}(s) ds = \int_T S(V,z)\phi_\mathfrak{h}(\mathbf{x}) d\mathbf{x}$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

$\mathcal{V}_{\mathfrak{h}} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_{\mathfrak{h}}\,,\ v_{|T} \in \mathbb{P}^d(T)\}$



Approximate solution : $V_{\mathfrak{h}}(\mathbf{x},t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

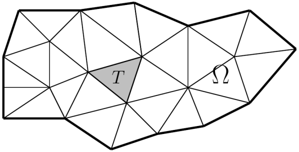
Through a semi-discrete formulation (1)

• $V \to V_{\mathfrak{h}}$

$$\int_T \frac{\partial}{\partial t}\Big(\sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\Big)\phi_{\mathfrak{h}}(\mathbf{x})d\mathbf{x} - \int_T H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\nabla\phi_{\mathfrak{h}}(\mathbf{x})d\mathbf{x} + \int_{\partial T} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}\phi_{\mathfrak{h}}(s)ds = \int_T S(V_{\mathfrak{h}}, z_{\mathfrak{h}})\phi_{\mathfrak{h}}(\mathbf{x})d\mathbf{x}$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on T of degree at most $d\}$.

$$\mathcal{V}_{\mathfrak{h}} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_{\mathfrak{h}}\,,\, v_{|T} \in \mathbb{P}^d(T)\}$$



Approximate solution : $V_{\mathfrak{h}}(\mathbf{x},t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Through a semi-discrete formulation (2)

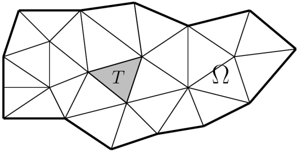
- $V \to V_{\mathfrak{h}}$
- $\phi_{\mathfrak{h}} \to \theta_j$

$$\int_T \frac{\partial}{\partial t}\left(\sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\right)\theta_j(\mathbf{x})d\mathbf{x} - \int_T H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\nabla\theta_j(\mathbf{x})d\mathbf{x} +$$

$$\int_{\partial T} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}\theta_j(s)ds = \int_T S(V_{\mathfrak{h}}, z_{\mathfrak{h}})\theta_j(\mathbf{x})d\mathbf{x}$$

$\mathbb{P}^d(T) := \{$2 variables polynomials on $T$of degree at most $d\}$.

$\mathcal{V}_{\mathfrak{h}} := \{v \in L^2(\Omega) \,|\, \forall T \in \mathcal{T}_{\mathfrak{h}}\,,\, v_{|T} \in \mathbb{P}^d(T)\}$



Approximate solution : $V_{\mathfrak{h}}(\mathbf{x}, t) = \sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\,, \quad \forall \mathbf{x} \in T\,.$

Through a semi-discrete formulation

- $V \to V_{\mathfrak{h}}$
- $\phi_{\mathfrak{h}} \to \theta_j$

$$\int_T \frac{\partial}{\partial t}\left(\sum_{l=1}^{N_d} V_l(t)\theta_l(\mathbf{x})\right)\theta_j(\mathbf{x})d\mathbf{x} - \int_T H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\nabla\theta_j(\mathbf{x})d\mathbf{x} +$$

$$\sum_{k=1}^{3}\int_{\Gamma_{ij(k)}} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)}\theta_j(s)ds = \int_T S(V_{\mathfrak{h}}, z_{\mathfrak{h}})\theta_j(\mathbf{x})d\mathbf{x}$$

Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

Contributions on the edges

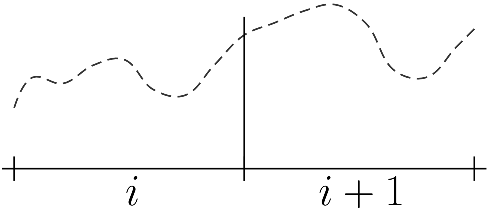
$$\int_{\Gamma_{ij(k)}} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$

Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

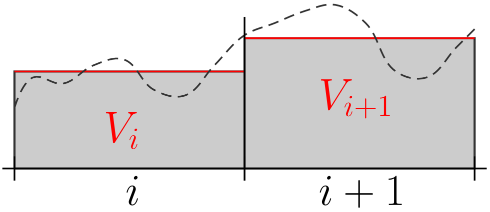
$$H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$



Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$

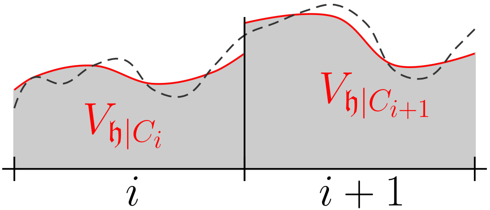


VF

Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$



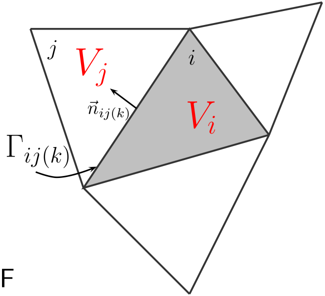
DG

Numerical fluxes

Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$

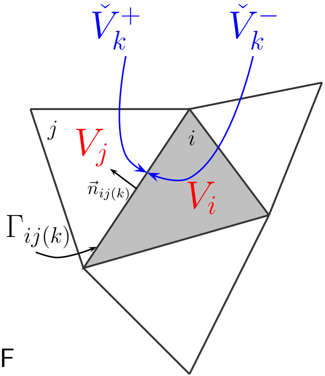


VF

Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$

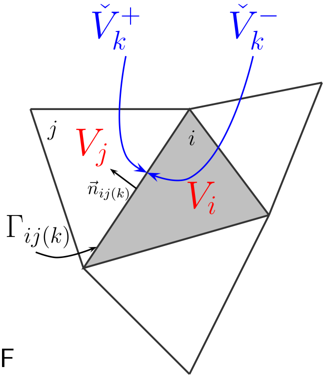


VF

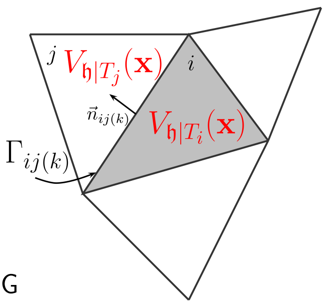
Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_{\mathfrak{h}}, z_{\mathfrak{h}}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$



VF

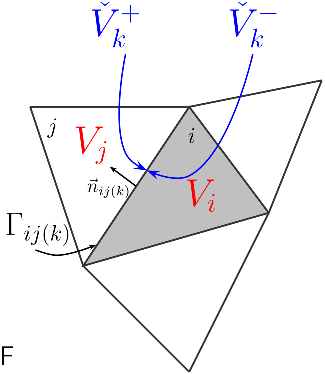


DG

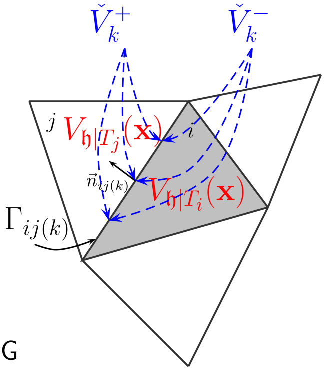
Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$



VF



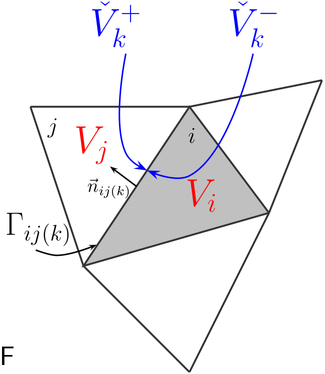
DG

Numerical issues 1
Arnaud Duran
Shallow Water Equations
Finite Volume approach : Stability issues
dG extension
dG method : generalities
Numerical fluxes
Preservation of the water depth positivity
Shocks
Extension to dispersive equations
Perspectives

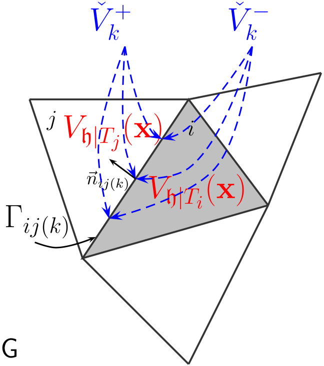
Contributions on the edges

$$\int_{\Gamma_{ij(k)}} H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)}\theta_j(s)ds\,.$$

$$H(V_\mathfrak{h}, z_\mathfrak{h}).\vec{n}_{ij(k)} \approx \mathcal{H}_{ij(k)} = \mathcal{H}(\check{V}_k^-, \check{V}_k^+, \check{z}_k, \check{z}_k, \vec{n}_{ij(k)})\,.$$



VF



DG

$\rightarrow$ Preservation of the motionless steady states.

dG schemes and maximum principle

- [X. Zhang, C.-W. Shu, 2010] Maximum-principle-satisfying high order schemes - 1d and 2d structured meshes
- [Y. Xing, X. Zhang, C.-W. Shu, 2010] Application to 1d SW
- [Y. Xing, X. Zhang, 2013] Extension to triangular meshes

The method

- relies on special quadrature rule such that :

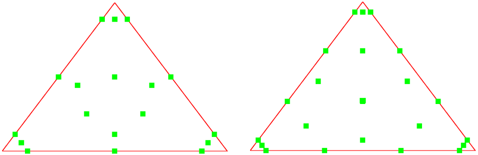


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

The method

- relies on special quadrature rule such that :
  - ▷ it includes faces nodes

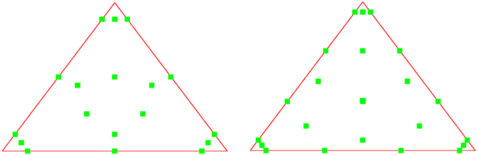


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

The method

- relies on special quadrature rule such that :
  - ▷ it includes faces nodes
  - ▷ the quadrature rule is exact for integration of $\eta_{\mathfrak{h}}$ on each element

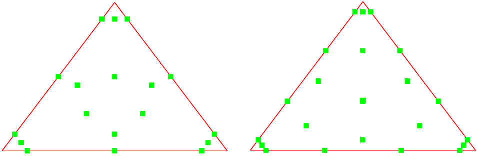


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

The method

- relies on special quadrature rule such that :
  - ▷ it includes faces nodes
  - ▷ the quadrature rule is exact for integration of $\eta_{\mathfrak{h}}$ on each element
  - ▷ all the quadrature weights are positive

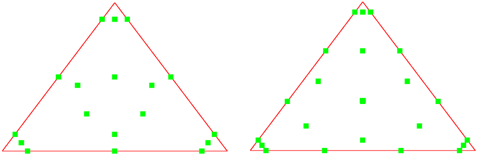


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

The method

- relies on special quadrature rule such that :
  - ▷ it includes faces nodes
  - ▷ the quadrature rule is exact for integration of $\eta_{\mathfrak{h}}$ on each element
  - ▷ all the quadrature weights are positive

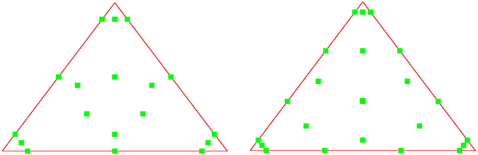


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

- requires the positivity of the water height at each node.
  ▷ A priori limitation step.

The method

- relies on special quadrature rule such that :
  - ▷ it includes faces nodes
  - ▷ the quadrature rule is exact for integration of $\eta_{\mathfrak{h}}$ on each element
  - ▷ all the quadrature weights are positive

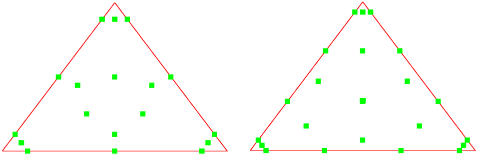
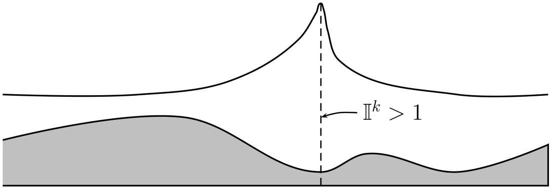


Figure: Nodes locations for the special quadrature - $\mathbb{P}^2$ and $\mathbb{P}^3$

- requires the positivity of the water height at each node.
  ▷ A priori limitation step.
- reduces to the study of a convex combination of **first order Finite Volume schemes**.

Shock treatment

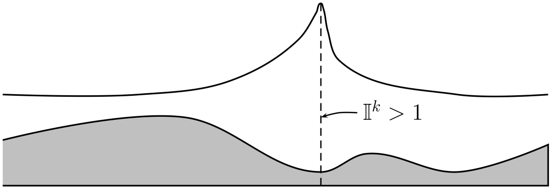
- High gradients detection.
  ▷ [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*.



- Restriction to $\mathbb{P}^1$ and application of a limiter on the discriminated elements ($\mathbb{I}^k > 1$).
  ▷ [B. Cockburn, C.-W. Shu, 1998] *The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V*
  ▷ [A. Burbeau, P. Sagaut, C.-H. Bruneau, 2001] Improvements on diffusive properties

- High gradients detection.

  ▷ [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*.



- Restriction to $\mathbb{P}^1$ and application of a limiter on the discriminated elements ($\mathbb{I}^k > 1$).

  ▷ [B. Cockburn, C.-W. Shu, 1998] *The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V*

  ▷ [A. Burbeau, P. Sagaut, C.-H. Bruneau, 2001] Improvements on diffusive properties

Outline

Objective

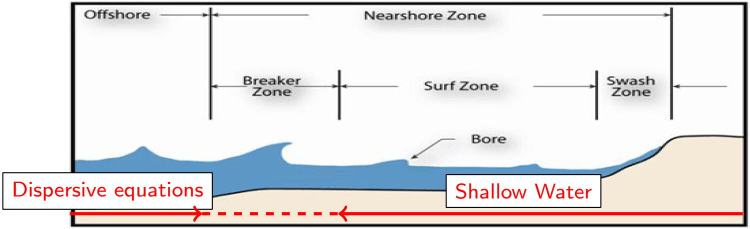
Extend the range of applicability of the computations at coast.

▷ Describe the non-linearities **before** the breaking point.

▷ Dispersive equations : $O(\mu^2)$

Shallow Water equations : $O(\mu)$.

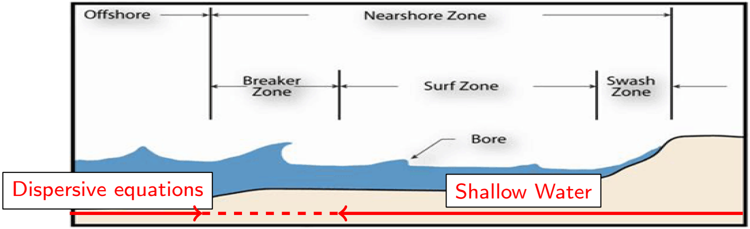
$$\text{Shallowness parameter} : \mu = \frac{h_0^2}{\lambda_0^2} << 1 .$$



Numerical issues 1
Arnaud Duran
Shallow Water Equations
Finite Volume approach : Stability issues
dG extension
Extension to dispersive equations
Motivations
The physical model
Reformulation of the system
High order derivatives
Handling breaking waves
Numerical validations
Perspectives

Numerical issues

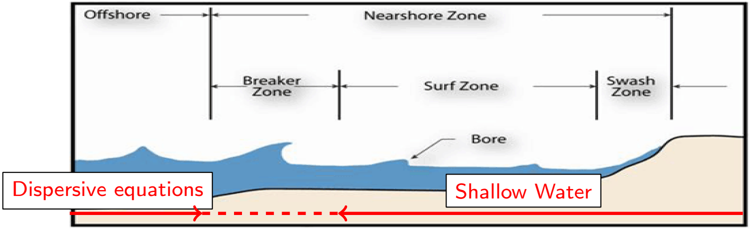
- ▷ Non conservative terms, high order derivatives, wave-breaking, non linearities.
- ▷ Maintain the stability of the method (positivity, well balancing), even on unstructured environments.



Numerical issues

▷ Non conservative terms, high order derivatives, wave-breaking, non linearities.

▷ Maintain the stability of the method (positivity, well balancing), even on unstructured environments.



State of the art

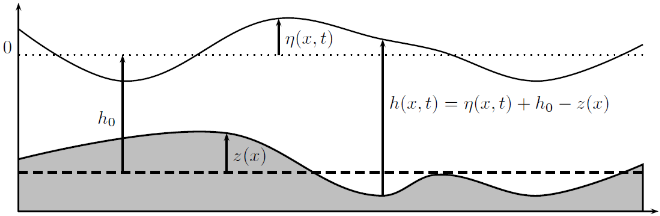
▷ 1d works : [Antunes Do Carmo *et al*] (FD, 1993), [Cienfuegos *et al*] (FV, 2006), [Dutykh *et al*] (FV, 2013), [Panda *et al*] (dG, 2014), [AD, Marche] (dG, 2015)

▷ 2d works : [Marche, Lannes] (Hybrid FV/FD, cartésien, 2015), [Popinet] (Hybrid FV/FD, cartésien, 2015)

▷ Unstructured meshes : Weakly non linear models (Boussinesq - type). [Kazolea, Delis, Synolakis] (FV, 2014), [Filippini, Kazolea, Ricchiuto] (Hybrid FV/FE, 2016)

Model presentation

- [P. Bonneton *et al*, 2011] 1d derivation and optimized model. Hybrid method.
- [F. Chazel, D. Lannes, F. Marche, 2011] 3 parameters model
- [M. Tissier *et al*, 2012] Wave breaking issues
- [D. Lannes, F. Marche, 2015] *A new class of fully nonlinear and weakly dispersive Green-Naghdi models for efficient 2D simulations*

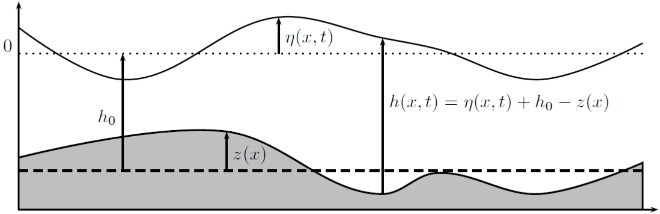


Revoke the time dependency

$$\begin{cases} \partial_t \eta + \partial_x(hu) = 0\,, \\ \left[1 + \alpha \mathfrak{T}[h_b]\right]\left(\partial_t hu + \partial_x(hu^2) + \frac{\alpha-1}{\alpha} gh\partial_x\eta\right) + \frac{1}{\alpha} gh\partial_x \eta \\ \qquad + h\left(\mathcal{Q}_1(u) + g\mathcal{Q}_2(\eta)\right) + g\mathcal{Q}_3\left(\left[1+\alpha\mathfrak{T}[h_b]\right]^{-1}(gh\partial_x\eta)\right) = 0\,. \end{cases}$$

Model presentation

▷ [D. Lannes, F. Marche, 2015] *A new class of fully nonlinear and weakly dispersive Green-Naghdi models for efficient 2D simulations*



Revoke the time dependency

$$\begin{cases} \partial_t \eta + \partial_x (hu) = 0\,, \\ \left[1 + \alpha \mathfrak{T}[h_b]\right] \left(\partial_t hu + \partial_x (hu^2) + \frac{\alpha - 1}{\alpha} g h \partial_x \eta \right) + \frac{1}{\alpha} g h \partial_x \eta \\ \qquad + h \left(\mathcal{Q}_1(u) + g \mathcal{Q}_2(\eta) \right) + g \mathcal{Q}_3 \left(\left[1 + \alpha \mathfrak{T}[h_b]\right]^{-1} (g h \partial_x \eta) \right) = 0\,. \end{cases}$$

▷ $\mathfrak{T}[h] w = -\frac{h^3}{3} \partial_x{}^2 \left(\frac{w}{h} \right) - h^2 \partial_x h \partial_x \left(\frac{w}{h} \right) \quad , \quad h_b = h_0 - z\,,$

▷ $\mathcal{Q}_{i=1,2,3}$: non linear, non conservative terms with second order derivatives.

▷ 2d version : "**diagonal**" sytem : no coupling between u and v !

Numerical issues 1

Arnaud Duran

Shallow Water Equations

Finite Volume approach : Stability issues

dG extension

Extension to dispersive equations

Motivations

The physical model

Reformulation of the system

High order derivatives

Handling breaking waves

Numerical validations

Perspectives

A convenient formulation

$$\underbrace{\frac{\partial}{\partial t}U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}}$$

▷ **Shallow Water** equations :

$$U = \begin{pmatrix} h \\ hu \end{pmatrix} \quad , \quad G(U) = \begin{pmatrix} hu \\ \frac{1}{2}gh^2 + hu^2 \end{pmatrix} \quad , \quad B(U) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix} .$$

A convenient formulation

$$\underbrace{\frac{\partial}{\partial t}U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}} + \underbrace{\mathfrak{D}(U,z)}_{\text{Dispersive terms}}$$

▷ **Shallow Water** equations :

$$U = \begin{pmatrix} h \\ hu \end{pmatrix} \quad , \quad G(U) = \begin{pmatrix} hu \\ \frac{1}{2}gh^2 + hu^2 \end{pmatrix} \quad , \quad B(U) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix} .$$

▷ **Dispersive** terms :

$$\mathfrak{D}(V,z) = \begin{pmatrix} 0 \\ \mathfrak{D}_{hu}(V,z) \end{pmatrix} , \text{ with}$$

$$\mathfrak{D}_{hu}(V,z) = \left[1 + \alpha\mathfrak{T}[h_b]\right]^{-1}\left(\frac{1}{\alpha}gh\partial_x\eta + h\big(\mathcal{Q}_1(u) + g\,\mathcal{Q}_2(\eta)\big)\right.$$
$$\left. + g\,\mathcal{Q}_3\left(\left[1 + \alpha\mathfrak{T}[h_b]\right]^{-1}(gh\partial_x\eta)\right)\right) - \frac{1}{\alpha}gh\partial_x\eta .$$

A convenient formulation

$$\underbrace{\frac{\partial}{\partial t}U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}} + \underbrace{\color{red}{\mathfrak{D}(U,z)}}_{\text{Dispersive terms}}$$

- Hyperbolic part : ok
- Dispersive part :

$$D_{\mathfrak{h}}(x,t) = \sum_{l=1}^{N_d} D_l(t)\theta_l(x)\,, \ x \in C_i\,.$$

  - Well balancing and robustness ,
  - Treatment of the second order derivatives .

A convenient formulation

$$\underbrace{\frac{\partial}{\partial t}U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}} + \underbrace{\mathfrak{D}(U,z)}_{\text{Dispersive terms}}$$

- Hyperbolic part : ok
- Dispersive part :

$$D_{\mathfrak{h}}(x,t) = \sum_{l=1}^{N_d} D_l(t)\theta_l(x)\,, \ x \in C_i\,.$$

- Well balancing and robustness ,
- Treatment of the second order derivatives .

A convenient formulation

$$\underbrace{\frac{\partial}{\partial t}U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}} + \underbrace{\color{red}{\mathfrak{D}(U,z)}}_{\text{Dispersive terms}}$$

- Hyperbolic part : ok
- Dispersive part :

$$D_{\mathfrak{h}}(x,t) = \sum_{l=1}^{N_d} D_l(t)\theta_l(x)\,,\ x \in C_i\,.$$

- Well balancing and robustness ,
- Treatment of the second order derivatives .

A convenient formulation

$$\underbrace{\frac{\partial}{\partial t}U + \partial_x G(U) = B(U,z)}_{\text{Shallow Water}} + \underbrace{\color{red}{\mathfrak{D}(U,z)}}_{\text{Dispersive terms}}$$

- Hyperbolic part : ok
- Dispersive part :

$$D_{\mathfrak{h}}(x,t) = \sum_{l=1}^{N_d} D_l(t)\theta_l(x)\,, \;\; x \in C_i\,.$$

  - Well balancing and robustness ,
  - Treatment of the second order derivatives .

Simplified case : $T = \partial_x{}^2$

Consider the second order ODE :

$$f - \partial_x^2 u = 0\,. \tag{2}$$

(2) reduces to a coupled system of first order equations.

$$f + \partial_x v = 0 \quad , \quad v + \partial_x u = 0\,.$$

Weak formulation

$$\int_{x_i^l}^{x_i^r} f\phi_{\mathfrak{h}} - \int_{x_i^l}^{x_i^r} v\phi_{\mathfrak{h}}' + \widehat{v}_r\phi_{\mathfrak{h}}(x_i^r) - \widehat{v}_l\phi_{\mathfrak{h}}(x_i^l) = 0\,,$$

$$\int_{x_i^l}^{x_i^r} v\phi_{\mathfrak{h}} - \int_{x_i^l}^{x_i^r} u\phi_{\mathfrak{h}}' + \widehat{u}_r\phi_{\mathfrak{h}}(x_i^r) - \widehat{u}_l\phi_{\mathfrak{h}}(x_i^l) = 0\,.$$

Simplified case : $T = {\partial_x}^2$

Consider the second order ODE :

$$f - \partial_x^2 u = 0 \,. \qquad (2)$$

(2) reduces to a coupled system of first order equations.

$$f + \partial_x v = 0 \quad , \quad v + \partial_x u = 0 \,.$$

Weak formulation

$$\int_{x_i^l}^{x_i^r} f\phi_{\mathfrak{h}} - \int_{x_i^l}^{x_i^r} v\phi'_{\mathfrak{h}} + \widehat{v}_r\phi_{\mathfrak{h}}(x_i^r) - \widehat{v}_l\phi_{\mathfrak{h}}(x_i^l) = 0 \,,$$

$$\int_{x_i^l}^{x_i^r} v\phi_{\mathfrak{h}} - \int_{x_i^l}^{x_i^r} u\phi'_{\mathfrak{h}} + \widehat{u}_r\phi_{\mathfrak{h}}(x_i^r) - \widehat{u}_l\phi_{\mathfrak{h}}(x_i^l) = 0 \,.$$

LDG schemes :

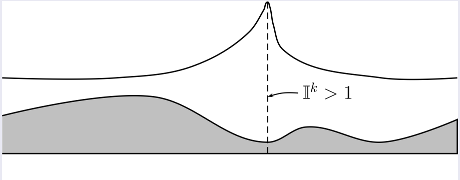
[B. Cockburn, C.-W. Shu, 1998] *The Local Discontinuous Galerkin method for time-dependent convection-diffusion systems* .

Protocol : At each time step

- Detection : evaluation of $\mathbb{I}^k$ on each cell k.
  [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*
- Determination of the breaking area.
- Switching strategy
  - Suppress the dispersive terms on the targeted area.
  - Application of limiter to treat the hyperbolic part (Shallow Water).

Protocol : At each time step

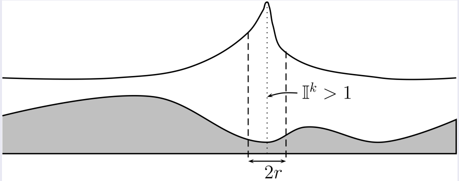
- Detection : evaluation of $\mathbb{I}^k$ on each cell k.
  [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*
- Determination of the breaking area.



- Switching strategy
  - Suppress the dispersive terms on the targeted area.
  - Application of limiter to treat the hyperbolic part (Shallow Water).

Protocol : At each time step

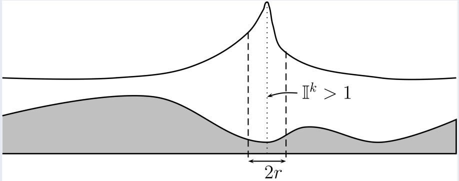
- Detection : evaluation of $\mathbb{I}^k$ on each cell k.
  [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*
- Determination of the breaking area.



- Switching strategy
  - Suppress the dispersive terms on the targeted area.
  - Application of limiter to treat the hyperbolic part (Shallow Water).

Protocol : At each time step

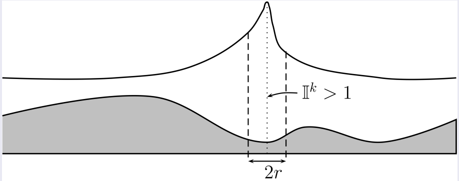
- Detection : evaluation of $\mathbb{I}^k$ on each cell k.
  [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*
- Determination of the breaking area.



- Switching strategy
  - Suppress the dispersive terms on the targeted area.
  - Application of limiter to treat the hyperbolic part (Shallow Water).

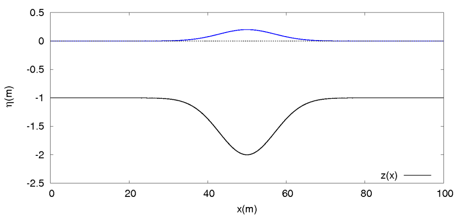
Protocol : At each time step

- Detection : evaluation of $\mathbb{I}^k$ on each cell k.
  [L. Krivodonova *et al*, 2004] *Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws*
- Determination of the breaking area.



- Switching strategy
  - Suppress the dispersive terms on the targeted area.
  - Application of limiter to treat the hyperbolic part (Shallow Water).

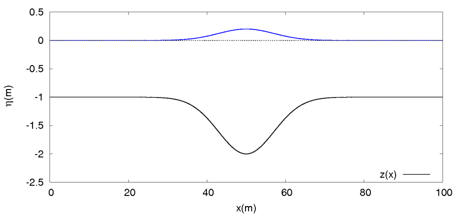
Profiles



Convergence rates

N	N_e 20	40	80	160	320	order
1	2.5e-1	4.2e-2	1.0e-3	2.8e-3	9.6e-4	1.9
2	7.5e-2	7.5e-3	6.2e-4	6.5e-5	7.7e-6	3.2
3	4.5e-3	3.0e-4	1.7e-5	9.4e-7	5.7e-8	4.0
4	7.0e-4	1.6e-5	4.6e-7	1.4e-8	4.4e-10	5.1
5	6.1e-5	7.6e-7	1.0e-8	1.6e-10	3.1e-12	6.1

Profiles



Convergence rates

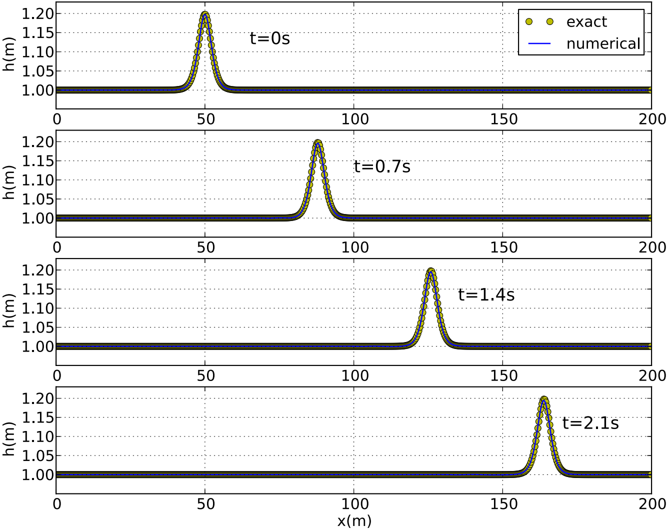
N	N_e 20	40	80	160	320	order
1	2.5e-1	4.2e-2	1.0e-3	2.8e-3	9.6e-4	1.9
2	7.5e-2	7.5e-3	6.2e-4	6.5e-5	7.7e-6	3.2
3	4.5e-3	3.0e-4	1.7e-5	9.4e-7	5.7e-8	4.0
4	7.0e-4	1.6e-5	4.6e-7	1.4e-8	4.4e-10	5.1
5	6.1e-5	7.6e-7	1.0e-8	1.6e-10	3.1e-12	6.1

CPU time

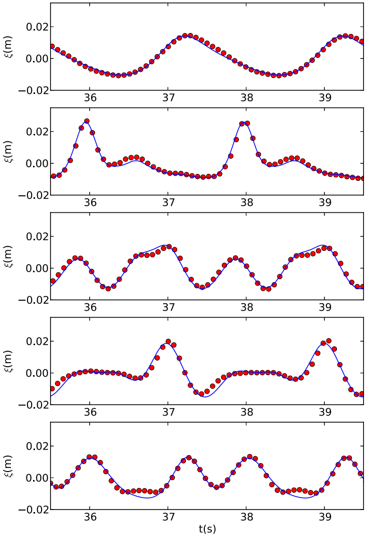
Evolution of the ratio $\tau = \rho_o / \rho_c$.
ρ : mean iteration time (based on 1000 iterations).

	N_e					
N	1000	2000	3000	4000	5000	6000
1	3.23	3.21	3.04	2.96	2.94	2.90
2	4.09	4.27	4.14	4.01	3.90	3.84
3	5.32	5.11	5.03	4.97	4.91	4.87
4	6.01	5.77	5.67	5.63	5.51	5.55
5	6.66	6.38	6.32	6.30	6.26	6.16
6	7.15	6.99	7.05	6.97	6.86	6.54

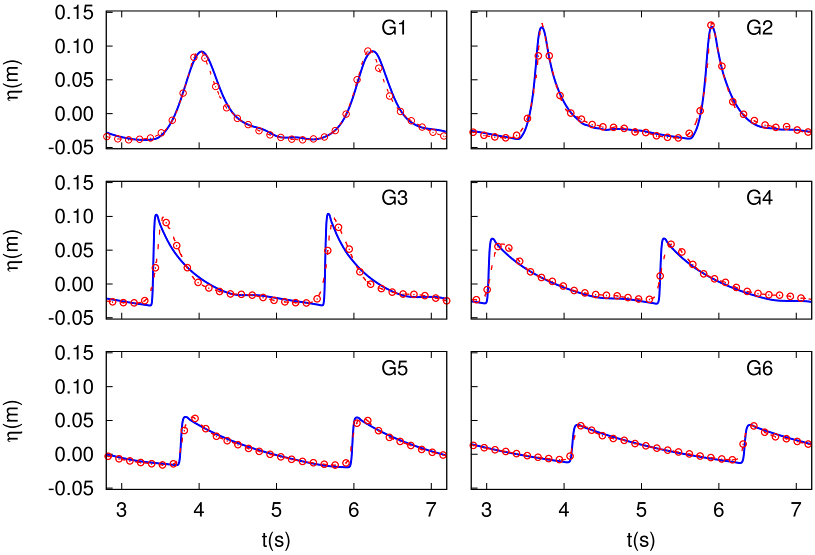
Propagation of solitary wave



Propagation of highly dispersive waves



Wave breaking : Cox test case



Outline

1. Finite Volume approach : Stability issues
2. dG extension
3. Extension to dispersive equations
4. Perspectives

Boundary conditions

→ Numerical methods.
→ Theoretical investigations.

Boundary conditions

→ Numerical methods.
→ Theoretical investigations.

Handling breaking waves

→ Coupling SW/GN, smoothness criteria.
→ Modelization issues.

• [S. Gavrilyuk, V. Liapidevskii, A. Chesnokov, 2016] *Spilling breakers : applications to Favre waves and to the shoaling and the breaking of the solitary wave*.

• Works of S. Gavrilyuk et G. Richard.

Perspectives

Boundary conditions

→ Numerical methods.
→ Theoretical investigations.

Handling breaking waves

→ Coupling SW/GN, smoothness criteria.
→ Modelization issues.

- [S. Gavrilyuk, V. Liapidevskii, A. Chesnokov, 2016] *Spilling breakers : applications to Favre waves and to the shoaling and the breaking of the solitary wave*.
- Works of S. Gavrilyuk et G. Richard.

Numerical treatment of the non-hydrostatic terms

→ Weak formulations.
→ Numerical exploration (linear solvers, matrix storage, re-numbering).
→ Alternative approaches (quadrature rules, Finite Volume methods).

- [R. Eymard,T. Gallouët, R. Herbin, ...] FV for non-linear elliptic problems.

Thank you !